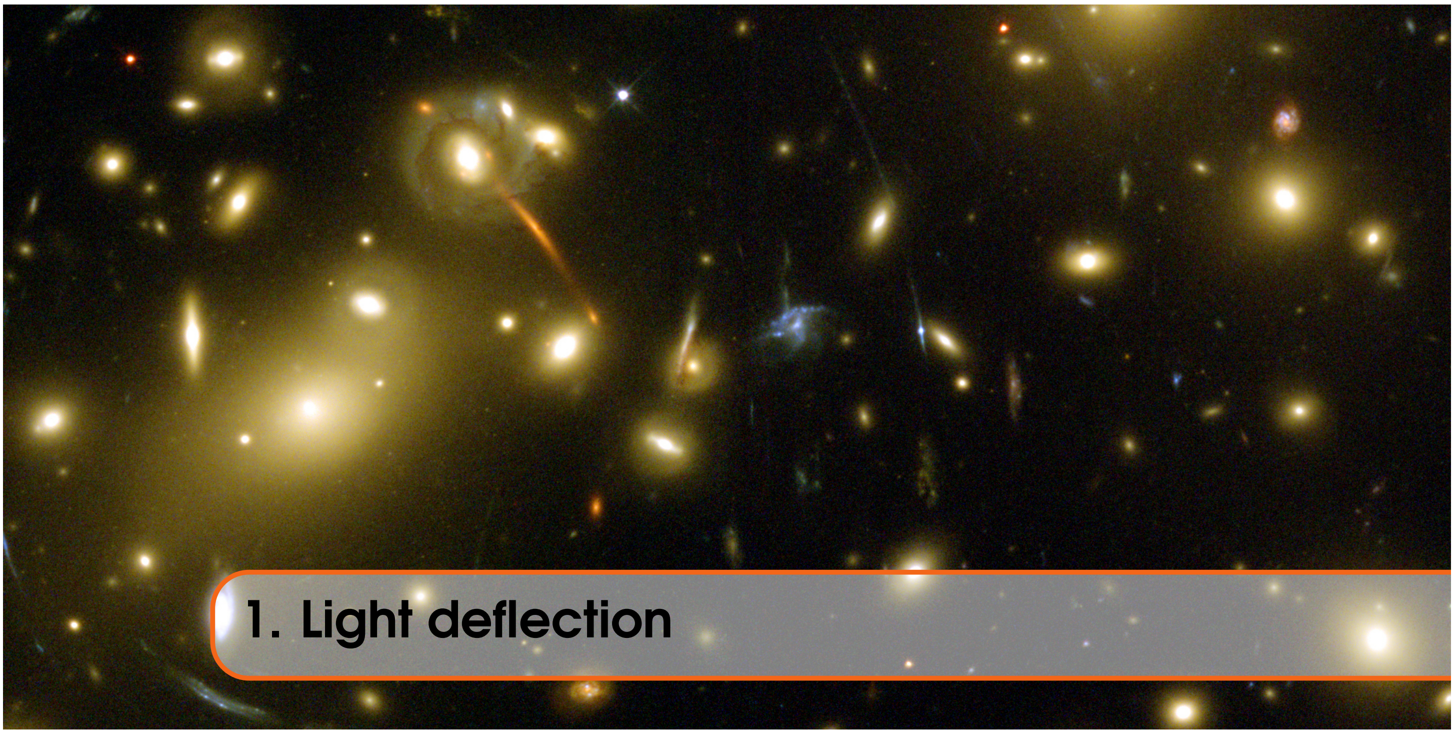


1. Light deflection

1.1 Deflection of a light corpuscle

The idea that light could be bent by gravity was mentioned by Isaac Newton in a note at the end of *Optiks*, pusblished in 1704. Further calculations were made about a century later by the German astronomer Johann Georg Von Soldner (1776-1833), who ended up quantifying that the deflection of a photon grazing the surface of the sun would amount to about 0.9”.

What were the assumptions under which this result was obtain? Well, we should first of all introduce the framework within which the idea was proposed. This is the so called “Corpuscolar Theory of Light”, which assumes that photons are not mass-less.

In this framework, the derivation of the deflection angle of a photon by a mass M is rather straightforward. It can be done in many ways, but we re-propose here a simple calculation by Victor J. Stenger (2013), which is based on three ingredients:

- Newton’s law of gravity;
- Newton’s equivalence principle;
- Einstein’s special relativity.

Newton’s law of gravity says that the gravitational force between two bodies with masses m and M is

$$\vec{F} = \frac{GmM}{r^3}\vec{r}\,, \tag{1.1}$$

where r is the distance between the bodies, and G is the gravitational constant.

In its *weak* form, Newton’s equivalence principle states that

$$\vec{F} = m\vec{a} \tag{1.2}$$

where a is the acceleration. The gravitational mass m in Eq. 1.1 equals the inertial mass m in Eq. 1.2.

From Einstein’s special relativity, we have that the inertial mass of a photon with energy E is E/c^2, where c is the speed of light.

Let assume that a photon with initial momentum $\vec{p}$ grazes the surface of the Sun, as shown in Fig. 1.1.1. The photon travels along the x-axis, while the y-axis was chosen to pass through the center of the sun, whose mass is M and whose radius is R. Let a be the impact parameter of the photon, i.e. the minimal distance at with the un-deflected trajectory of the photon passes from the Sun center. When the photon is at the position (x,y), the distance from the Sun is

$$r = \sqrt{x^2 + (a-y)^2}\ . \tag{1.3}$$

Let's assume that the moment of the photon does not change significantly along its path. The

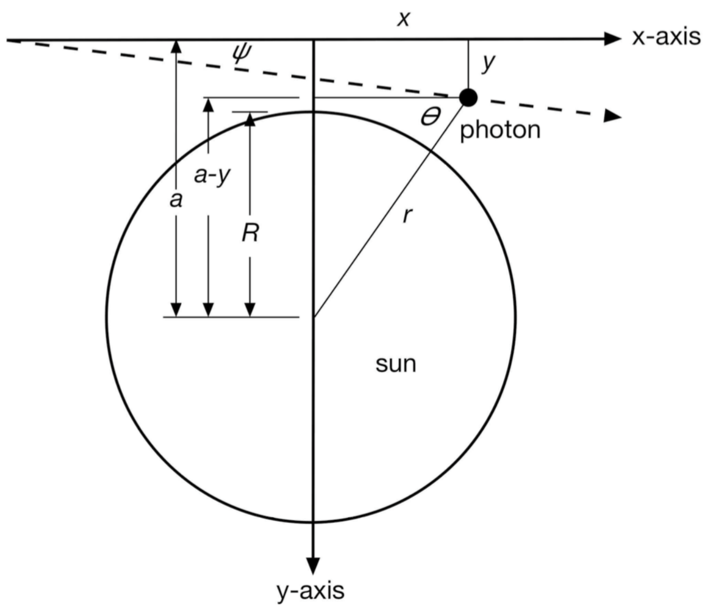


Figure 1.1.1: *Schematic view of a photon grazing the surface of the Sun (from V. J. Stenger, 2013).*

components of the gravitational force acting on the photon are

$$F_x = \frac{dp}{dt}\cos\theta = \frac{GMp}{c[x^2+(a-y)^2]}\cos\theta = \frac{GMp}{c}\frac{x}{[x^2+(a-y)^2]^{3/2}}\ , \tag{1.4}$$

$$F_y = \frac{dp}{dt}\sin\theta = \frac{GMp}{c[x^2+(a-y)^2]}\sin\theta = \frac{GMp}{c}\frac{a-y}{[x^2+(a-y)^2]^{3/2}}\ . \tag{1.5}$$

$$\tag{1.6}$$

Now, let's assume that $x = ct$. We can then write:

$$\frac{dp_i}{dt} = \frac{dp_i}{dx}\frac{dx}{dt} = c\frac{dp_i}{dx}\ , \tag{1.7}$$

which allows to re-write Eqs. 1.6 as

$$\frac{dp_x}{dx} = \frac{GMp}{c^2}\frac{x}{[x^2+(a-y)^2]^{3/2}}\ , \tag{1.8}$$

$$\frac{dp_y}{dx} = \frac{GMp}{c^2}\frac{a-y}{[x^2+(a-y)^2]^{3/2}}\ . \tag{1.9}$$

$$\tag{1.10}$$

These equations allow us to calculate by how much does the momentum change along the x and the y axes as the x coordinate of the photon changes. Along the $x-$axis:

$$\begin{aligned}\Delta p_x &= \frac{GMp}{c^2}\int_{-\infty}^{\infty}\frac{GMp}{c^2}\frac{x}{[x^2+(a-y)^2]^{3/2}}dx \\ &= \frac{GMp}{c^2}[\log[(a-y)^2+x^2]]_{-\infty}^{+\infty}=0\,. \end{aligned} \tag{1.11}$$

Thus, the photon momentum is un-changed along the x-axis. On the contrary, along the y-axis, the photon momentum changes by

$$\begin{aligned}\Delta p_y &= \frac{GMp}{c^2}\int_{-\infty}^{\infty}\frac{GMp}{c^2}\frac{a-y}{[x^2+(a-y)^2]^{3/2}}dx \\ &= \frac{GMp}{c^2}[\tan^{-1}\frac{x}{a-y}]_{-\infty}^{+\infty}=\frac{2GMp}{c^2}\frac{1}{a-y}\,, \end{aligned} \tag{1.12}$$

which can be used to compute the deflection angle

$$\psi=\frac{\Delta p_y}{p}=\frac{2GM}{c^2}\frac{1}{a-y}\,. \tag{1.13}$$

If the photon impact parameter is $a-y=R_\odot$, Eq. 1.13 reduces to

$$\psi=\frac{\Delta p_y}{p}=\frac{2GM}{c^2R_\odot}\approx 0.875\text{''}\,, \tag{1.14}$$

when inserting $M=M_\odot=1.989\times10^{30}$ kg and $R_\odot=6.96\times10^{8}$ m. Thus, using Newtonian gravity and assuming that photons are light corpuscles, we obtain that a photon grazing the surface of the Sun is deflected by 0.875”. We will see shortly that this value is just half of what predicted by Einsten in the framework of his Theory of General Relativity.

1.2 Deflection of light according to General Relativity

1.2.1 Fermat principle and light deflection

Starting from the field equations of general relativity, light deflection can be calculated by studying geodesic curves. It turns out that light deflection can equivalently be described by Fermat’s principle, as in geometrical optics. This will be our starting point.

Exercise 1.1 — Derive the Snell’s law from Fermat principle. In its simplest form the Fermat’s principle says that light waves of a given frequency traverse the path between two points which takes the least time. The speed of light in a medium with refractive index n is c/n, where c is its speed in a vacuum. Thus, the time required for light to go some distance in such a medium is n times the time light takes to go the same distance in a vacuum.

Referring to Fig. 1.2.1, the time required for light to go from A to B becomes

$$t=[\{h_1^2+y^2\}^{1/2}+n\{h_2^2+(w-y)^2\}^{1/2}]/c.$$

We find the minimum time by differentiating t with respect to y and setting the result to zero, with the result that

$$\frac{y}{\{h_1^2+y^2\}^{1/2}}=n\frac{w-y}{\{h_2^2+(w-y)^2\}^{1/2}}.$$

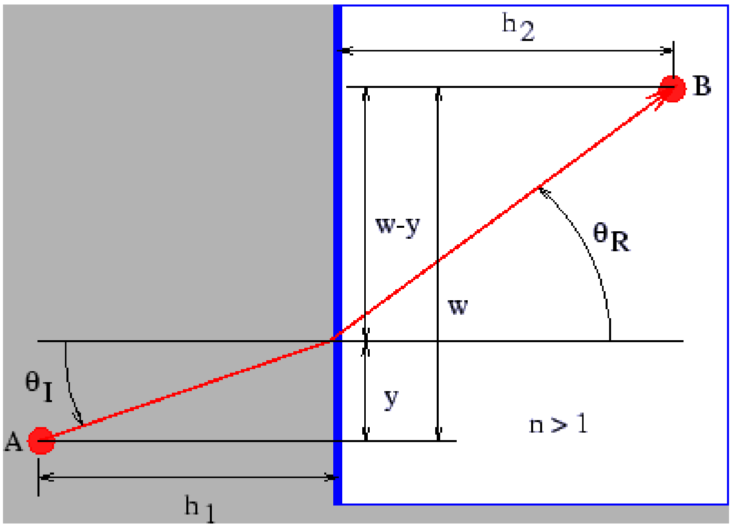


Figure 1.2.1: *Definition sketch for deriving Snell's law of refraction from Fermat's principle. The shaded area has refractive index $n > 1$*

However, we note that the left side of this equation is simply $\sin\theta_I$, while the right side is $n\sin\theta_R$, so that the minimum time condition reduces to

$$\sin\theta_I = n\sin\theta_R$$

We recognize this result as Snell's law.

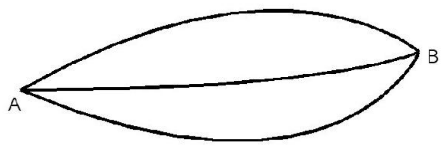
Taking inspiration from the Exercise above, we attempt to treat the deflection of light in a general relativity framework as a refraction problem. We need an refractive index n because Fermat's principle says that light will follow a path along which the travel time,

$$t_{travel} = \int \frac{n}{c} \mathrm{d}l \,, \tag{1.15}$$

will be extremal. As in geometrical optics, we thus search for a path, $\vec{x}(l)$, for which the variation

$$\delta \int_A^B n(\vec{x}(l))\mathrm{d}l = 0 \,, \tag{1.16}$$

where the starting point A and the end point B are kept fixed.



Deflection in the Minkowski's space-time

In order to find the refractive index, we make a first approximation: we assume that the lens is weak, and that it is small compared to the overall dimensions of the optical system composed of source, lens and observer. With "weak lens", we mean a lens whose Newtonian gravitational potential Φ is much smaller than c^2, $\Phi/c^2 \ll 1$. Note that this approximation is valid in virtually all cases of astrophysical interest. Consider for instance a galaxy cluster: its gravitational potential is $|\Phi| < 10^{-4}c^2 \ll c^2$. In addition, we also assume that the light deflection occurs in a region which is small enough that we can neglect the expansion of the universe.

In this case, the metric of unperturbed space-time is the Minkowski metric,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ,$$

whose line element is

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = (dx^0)^2 - (d\vec{x})^2 = c^2dt^2 - (d\vec{x})^2 . \tag{1.17}$$

Now, we consider a weak lens perturbing this metric, such that

$$\eta_{\mu\nu} \to g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1-\frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1-\frac{2\Phi}{c^2}) & \\ 0 & 0 & 0 & -(1-\frac{2\Phi}{c^2}) \end{pmatrix}$$

for which the line element becomes

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(1+\frac{2\Phi}{c^2}\right)c^2dt^2 - \left(1-\frac{2\Phi}{c^2}\right)(d\vec{x})^2 . \tag{1.18}$$

■ Example 1.1 — Schwarzschild metric in the weak field limit. Assuming a spherically symmetric and static potential, the Einstein's field equations can be solved to obtain the *Schwarzschild metric*. The line element is written in spherical coordinates as

$$ds^2 = \left(1-\frac{2GM}{Rc^2}\right)c^2dt^2 - \left(1-\frac{2GM}{Rc^2}\right)^{-1}dR^2 - R^2(\sin^2\theta d\phi^2 + d\theta^2) .$$

To obtain a simpler expression, it is convenient to introduce the new radial coordinate r, defined through

$$R = r\left(1+\frac{GM}{2rc^2}\right)^2$$

and the cartesian coordinates $x = r\sin\theta\cos\theta$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, so that $dl^2 = dx^2 + dy^2 + dz^2$. After some algebra, the metric can then be written in the form

$$ds^2 = \left(\frac{1-GM/2rc^2}{1+GM/2rc^2}\right)^2 c^2dt^2 - \left(1+\frac{GM}{2rc^2}\right)^4(dx^2+dy^2+dz^2) .$$

In the weak field limit, $\Phi/c^2 = -GM/rc^2 \ll 1$,

$$\begin{aligned} \left(\frac{1-GM/2rc^2}{1+GM/2rc^2}\right)^2 &\approx \left(1-\frac{GM}{2rc^2}\right)^4 \\ &\approx \left(1-\frac{2GM}{rc^2}\right) \\ &= \left(1+\frac{2\Phi}{c^2}\right) \end{aligned}$$

and

$$\left(1+\frac{GM}{2rc^2}\right)^4 \approx \left(1+2\frac{GM}{rc^2}\right) = \left(1-\frac{2\Phi}{c^2}\right) .$$

Therefore, the Schwarzschild metric in the weak field limit equals

$$\mathrm{d}s^2 = \left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2\Phi}{c^2}\right)\mathrm{d}l^2 ,$$

thus recovering Eq. 1.18. ■

Effective refractive index

Light propagates at zero eigentime, $\mathrm{d}s = 0$, from which we obtain

$$\left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 = \left(1-\frac{2\Phi}{c^2}\right)(\mathrm{d}\vec{x})^2 . \tag{1.19}$$

The light speed in the gravitational field is thus

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}} \approx c\left(1+\frac{2\Phi}{c^2}\right) , \tag{1.20}$$

where we have used that $\Phi/c^2 \ll 1$ by assumption. The refractive index is thus

$$n = c/c' = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1-\frac{2\Phi}{c^2} . \tag{1.21}$$

With $\Phi \leq 0$, $n \geq 1$, and the light speed c' is smaller than in absence of the gravitational potential.

Deflection angle

The refractive index n depends on the spatial coordinate $\vec{x}$ and perhaps also on time t. Let $\vec{x}(l)$ be a light path. Then, the light travel time is

$$t_{travel} \propto \int_A^B n[\vec{x}(l)]\mathrm{d}l , \tag{1.22}$$

and the light path follows from

$$\delta\int_A^B n[\vec{x}(l)]\mathrm{d}l = 0 . \tag{1.23}$$

This is a standard variational problem, which leads to the well known Euler equations. In our case we write

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right|\mathrm{d}\lambda , \tag{1.24}$$

with a curve parameter λ which is yet arbitrary, and find

$$\delta\int_{\lambda_A}^{\lambda_B}\mathrm{d}\lambda\, n[\vec{x}(\lambda)]\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0 \tag{1.25}$$

The expression

$$n[\vec{x}(\lambda)]\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \equiv L(\dot{\vec{x}},\vec{x},\lambda) \tag{1.26}$$

takes the role of the Lagrangian, with

$$\dot{\vec{x}} \equiv \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} \, . \tag{1.27}$$

Finally, we have

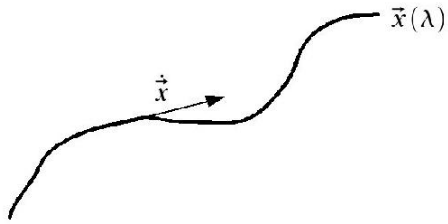
$$\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = |\dot{\vec{x}}| = (\dot{\vec{x}}^2)^{1/2} \, . \tag{1.28}$$

The Euler equation writes:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0 \, . \tag{1.29}$$

Now,

$$\frac{\partial L}{\partial \vec{x}} = |\dot{\vec{x}}|\frac{\partial n}{\partial \vec{x}} = (\vec{\nabla}n)|\dot{\vec{x}}| \, , \, \frac{\partial L}{\partial \dot{\vec{x}}} = n\frac{\dot{\vec{x}}}{|\dot{\vec{x}}|} \, . \tag{1.30}$$



Evidently, $\dot{\vec{x}}$ is a tangent vector to the light path , which we can assume to be normalized by a suitable choice for the curve parameter λ. We thus assume $|\dot{\vec{x}}| = 1$ and write $\vec{e} \equiv \dot{\vec{x}}$ for the unit tangent vector to the light path. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(n\vec{e}) - \vec{\nabla}n = 0 \, , \tag{1.31}$$

or

$$n\dot{\vec{e}} + \vec{e}\cdot[(\vec{\nabla}n)\dot{\vec{x}}] = \vec{\nabla}n \, ,$$

$$\Rightarrow n\dot{\vec{e}} = \vec{\nabla}n - \vec{e}(\vec{\nabla}n\cdot\vec{e}) \, . \tag{1.32}$$

The second term on the right hand side is the derivative along the light path, thus the whole right hand side is the gradient of n perpendicular to the light path. Thus

$$\dot{\vec{e}} = \frac{1}{n}\vec{\nabla}_{\perp}n = \vec{\nabla}_{\perp}\ln n \, . \tag{1.33}$$

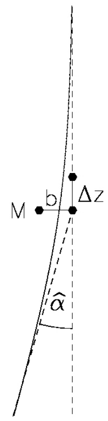
As $n = 1 - 2\Phi/c^2$ and $\Phi/c^2 \ll 1$, $\ln n \approx -2\Phi/c^2$, and

$$\dot{\vec{e}} \approx -\frac{2}{c^2}\vec{\nabla}_{\perp}\Phi \, . \tag{1.34}$$

The total deflection angle of the light path is now the integral over $-\dot{\vec{e}}$ along the light path,

$$\hat{\vec{\alpha}} = \frac{2}{c^2} \int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \mathrm{d}\lambda \;, \tag{1.35}$$

or, in other words, the integral over the "pull" of the gravitational potential perpendicular to the light path. Note that $\vec{\nabla}\Phi$ points away from the lens center, so $\hat{\vec{\alpha}}$ points in the same direction.



Born approximation

As it stands, the equation for $\hat{\vec{\alpha}}$ is not useful, as we would have to integrate over the actual light path. However, since $\Phi/c^2 \ll 1$, we expect the deflection angle to be small. Then, we can adopt the *Born approximation*, familiar from scattering theory, and integrate over the unperturbed light path.

Suppose, therefore, that a light ray starts out into $+\vec{e}_z$-direction and passes a lens at $z = 0$, with impact parameter b. The deflection angle is then given by

$$\hat{\vec{\alpha}}(b) = \frac{2}{c^2} \int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi \mathrm{d}z \tag{1.36}$$

■ **Example 1.2 — Deflection by a point mass.** If the lens is a point mass, then

$$\Phi = -\frac{GM}{r} \tag{1.37}$$

with $r = \sqrt{x^2+y^2+z^2} = \sqrt{b^2+z^2}$, $b = \sqrt{x^2+y^2}$ and

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix} . \tag{1.38}$$

The deflection angle is then

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{(b^2+z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2(b^2+z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} , \end{aligned} \tag{1.39}$$

with

$$\begin{pmatrix} x \\ y \end{pmatrix} = b \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{1.40}$$

Notice that $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius of a (point) mass M, thus

$$|\hat{\vec{\alpha}}| = \frac{4GM}{c^2 b} = 2\frac{R_s}{b} . \tag{1.41}$$

Also notice that $\hat{\vec{\alpha}}$ is linear in M, thus the superposition principle can be applied to compute the deflection angle of an ensemble of lenses. ■