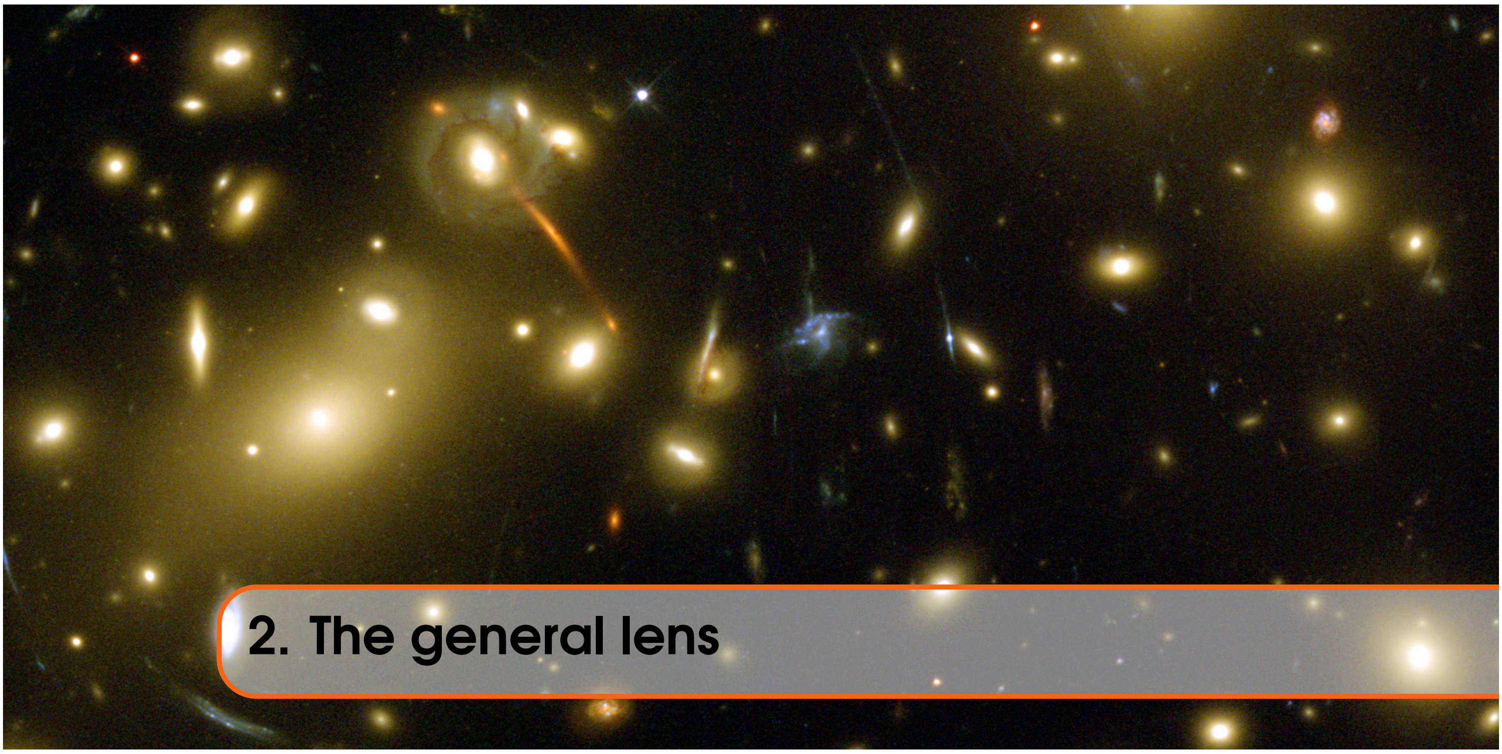

2. The general lens

2.1 Lens equation

Gravitational lensing is sensitive to the geometry of the universe. In particular, as in refractive phenomena, the amplitude of the lensing effects is heavily dependent on the distances between the observer, the lenses, and the sources. These are in turn related to the curvature and expansion rate of the universe, which suggests that gravitational lensing is indeed a powerful tool for cosmology.

Instrinsic and apparent source position

In this section, we seek a relationship between observed and intrinsic positions of a source in a gravitational lensing event. In absence of the lens, the light emitted by a distant source reaches an observer, who sees the source at a certain position on the sky, $\vec{\beta}$. This is the *intrinsic* position of the source. Instead, when photons are deflected by a gravitational lens, the observer collects them from a different direction, $\vec{\theta}$, which corresponds to the *apparent* (or *observed*) position of the source. It is common to refer to the apparent position of the source as to the *image* position.

In Fig. (2.1.1), we sketch a typical gravitational lens system. A mass is placed at redshift $z_{\rm L}$, corresponding to an angular diameter distance $D_{\rm L}$. This lens deflects the light rays coming from a source at redshift $z_{\rm S}$ (or angular distance $D_{\rm S}$). At the bottom of the diagram, an observer collects the photons from the distant source. The angular diameter distance between the lens and the source is $D_{\rm LS}$.

R The angular diameter distance D_A is defined as the ratio of an object's physical transverse size to its angular size (in radians). Therefore, it is used to convert angular separations in the sky to physical separations on the plane of the sources.

This distance does not increase indefinitely with redshift, but it peaks at $z \sim 1$ and then it turns over. Due to the expansion of the universe the angular diameter distance between z_1 and z_2 (with $z_2 > z_1$) is not found by subtracting the two individual angular diameter distances:

$$D_A(z_1, z_2) \neq D_A(z_2) - D_A(z_1) \tag{2.1}$$

except for those situations where the expansion of the universe can be neglected (i.e. for lenses and sources in our own galaxy). Hogg (1999) presents a concise summary on cosmological

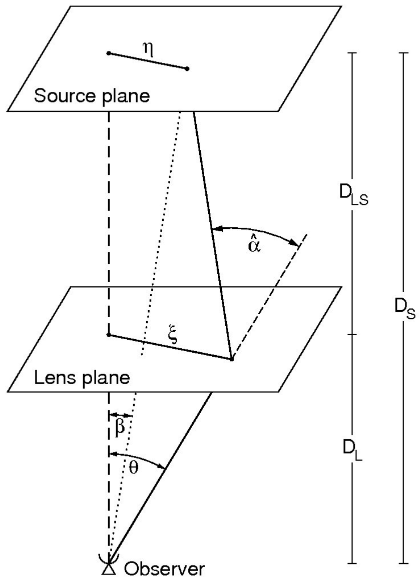

Figure 2.1.1: *Sketch of a typical gravitational lensing system (Figure from Bartelmann and Schneider (2001).*

distances. More in-depth discussions can be found in several cosmology books (see e.g. Weinberg, 1972).

Thin screen approximation

If the physical size of the lens is small compared to the distances D_L, D_LS, and D_S, the extension of the lens along the line-of-sight can be neglected in the calculation of the light deflection. We can assume that this occurs on a plane, called the *lens plane*.

R Given that the apparent position of the source, or image position, originates on this plane, the lens plane is often refereed as the *image plane*.

Similarly, we can assume that all photons emitted by the source originate from the same distance D_S, meaning that the source lies on a *source plane*. The approximation of the lens and of the source to planar distributions of mass and light, is called *thin screen approximation*.

Relating the intrinsic and apparent positions of the source

We first define an optical axis, indicated by the dashed line, perpendicular to the lens and source planes and passing through the observer. Then we measure the angular positions on the lens and on the source planes with respect to this reference direction.

Consider a source at the intrinsic angular position $\vec{\beta}$, which lies on the source plane at a distance $\vec{\eta} = \vec{\beta} D_\mathrm{S}$ from the optical axis. The source emits photons (we may now use the term “light rays") that impact the lens plane at $\vec{\xi} = \vec{\theta} D_\mathrm{L}$, are deflected by the angle $\hat{\vec{\alpha}}$, and finally reach the observer. The amplitude of the deflection is given by Eq. (1.36).

Due to the deflection, the observer receives the light coming from the source as if it was emitted at the apparent angular position $\vec{\theta}$. Note that we have used vectors to identify the source and image positions on the corresponding planes, either in the case of angular and physical positions.

If $\vec{\theta}$, $\vec{\beta}$ and $\hat{\vec{\alpha}}$ are small, the true position of the source and its observed position on the sky are related by a very simple relation, which can be readily obtained from the diagram in Fig. 2.1.1.

This relation is called the *lens equation* and is written as

$$\vec{\theta} D_S = \vec{\beta} D_S + \hat{\vec{\alpha}} D_{LS} \ , \tag{2.2}$$

where D_{LS} is the angular diameter distance between lens and source.

Defining the reduced deflection angle

$$\vec{\alpha}(\vec{\theta}) \equiv \frac{D_{LS}}{D_S} \hat{\vec{\alpha}}(\vec{\theta}) \ , \tag{2.3}$$

from Eq. (2.2), we obtain

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \ . \tag{2.4}$$

This equation, called *lens equation*, is apparently very simple. However, $\vec{\alpha}(\vec{\theta})$ can be a complicated function of $\vec{\theta}$, which implies that the equation can only be solved numerically in many cases.

It is very common and useful to write Eq. (2.2) in dimensionless form. This can be done by defining a length scale ξ_0 on the lens plane and a corresponding length scale $\eta_0 = \xi_0 D_S / D_L$ on the source plane. Then, we define the dimensionless vectors

$$\vec{x} \equiv \frac{\vec{\xi}}{\xi_0} \quad ; \quad \vec{y} \equiv \frac{\vec{\eta}}{\eta_0} \ , \tag{2.5}$$

as well as the scaled deflection angle

$$\vec{\alpha}(\vec{x}) = \frac{D_L D_{LS}}{\xi_0 D_S} \hat{\vec{\alpha}}(\xi_0 \vec{x}) \ . \tag{2.6}$$

Carrying out some substitutions, Eq. (2.2) can finally be written as

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \ . \tag{2.7}$$

Solving the lens equation

From Eqs.2.4 and 2.7, it is obvious that knowing the intrinsic position of the source and the deflection angle field $\alpha(\vec{\theta})$ of the lens, the positions of the image(s) can be be found by solving the lens equation for $\vec{\theta}$. As it will be discussed later on, this can be achieved analytically only for very simple lens mass distributions. Indeed, the equation is typically highly non-linear. When multiple solutions exist, the source is lensed into *multiple images*.

When observing a lens system, the intrinsic position of the source is unknown, while the position of its images can be measured. Then the source intrinsic position can be recovered by assuming a model for the mass distribution of the lens, i.e. by solving the lens equation for $\vec{\beta}$. This is a much easier task, because the lens equation is linear in $\vec{\beta}$: for each image there is a unique solution. Thus, if multiple images of the same source are identified, and the lens mass model is correct, the same solution of the lens equation should be found for all images.

2.2 Lensing potential

An extended distribution of matter is characterized by its *effective lensing potential*, obtained by projecting the three-dimensional Newtonian potential on the lens plane and by properly rescaling it:

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_L D_S} \frac{2}{c^2} \int \Phi(D_L \vec{\theta}, z) \mathrm{d}z \ . \tag{2.8}$$

The lensing potential satisfies two important properties:

1. **the gradient of $\hat{\Psi}$ is the reduced deflection angle**:

$$\vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) = \vec{\alpha}(\vec{\theta}) \ . \tag{2.9}$$

Indeed, by taking the gradient of the lensing potential we obtain:

$$\begin{aligned} \vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) &= D_\mathrm{L} \vec{\nabla}_\perp \hat{\Psi} = \vec{\nabla}_\perp \left(\frac{D_\mathrm{LS}}{D_\mathrm{S}} \frac{2}{c^2} \int \hat{\Phi}(\vec{\theta}, z) \mathrm{d}z \right) \\ &= \frac{D_\mathrm{LS}}{D_\mathrm{S}} \frac{2}{c^2} \int \vec{\nabla}_\perp \Phi(\vec{\theta}, z) \mathrm{d}z \\ &= \vec{\alpha}(\vec{\theta}) \end{aligned} \tag{2.10}$$

Note that, using the adimensional notation,

$$\vec{\nabla}_x = \frac{\xi_0}{D_\mathrm{L}} \vec{\nabla}_\theta \ . \tag{2.11}$$

We can see that

$$\vec{\nabla}_x \hat{\Psi} = \frac{\xi_0}{D_\mathrm{L}} \vec{\nabla}_\theta \hat{\Psi} = \frac{\xi_0}{D_\mathrm{L}} \vec{\alpha} \ . \tag{2.12}$$

By multiplying both sides of this equation by D_L^2/ξ_0^2, we obtain

$$\frac{D_\mathrm{L}^2}{\xi_0^2} \vec{\nabla}_x \hat{\Psi} = \frac{D_\mathrm{L}}{\xi_0} \vec{\alpha} \ . \tag{2.13}$$

This allows us to introduce the dimensionless counterpart of $\hat{\Psi}$:

$$\Psi = \frac{D_\mathrm{L}^2}{\xi_0^2} \hat{\Psi} \ . \tag{2.14}$$

Substituting Eq. 2.14 into Eq 2.13, we see that

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x}) \ . \tag{2.15}$$

2. **the Laplacian of $\hat{\Psi}$ is twice the *convergence* κ**:

$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta}) \ . \tag{2.16}$$

This is defined as a dimensionless surface density

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_\mathrm{cr}} \quad \text{with} \quad \Sigma_\mathrm{cr} = \frac{c^2}{4\pi G} \frac{D_\mathrm{S}}{D_\mathrm{L} D_\mathrm{LS}} \ , \tag{2.17}$$

where Σ_cr is called the *critical surface density*, a quantity which characterizes the lens system and which is a function of the angular diameter distances of lens and source.
Eq. 2.16 is derived from the Poisson equation,

$$\triangle \Phi = 4\pi G \rho \ . \tag{2.18}$$

The surface mass density is

$$\Sigma(\vec{\theta}) = \frac{1}{4\pi G} \int_{-\infty}^{+\infty} \triangle \Phi \mathrm{d}z \tag{2.19}$$

and

$$\kappa(\vec{\theta}) = \frac{1}{c^2} \frac{D_\mathrm{L} D_\mathrm{LS}}{D_\mathrm{S}} \int_{-\infty}^{+\infty} \triangle \Phi \mathrm{d}z \ . \tag{2.20}$$

Let us now introduce a two-dimensional Laplacian

$$\triangle_\theta = \frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2} = D_L^2\left(\frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2}\right) = D_L^2\left(\triangle - \frac{\partial^2}{\partial z^2}\right) , \tag{2.21}$$

which gives

$$\triangle\Phi = \frac{1}{D_L^2}\triangle_\theta\Phi + \frac{\partial^2\Phi}{\partial z^2} . \tag{2.22}$$

Inserting Eq. 2.22 into Eq. 2.20, we obtain

$$\kappa(\vec{\theta}) = \frac{1}{c^2}\frac{D_{LS}}{D_S D_L}\left[\triangle_\theta\int_{-\infty}^{+\infty}\Phi dz + D_L^2\int_{-\infty}^{+\infty}\frac{\partial^2\Phi}{\partial z^2}dz\right] . \tag{2.23}$$

If the lens is gravitationally bound, $\partial\Phi/\partial z = 0$ at its boundaries and the second term on the right hand side vanishes. From Eqs. 2.8 and 2.14, we find

$$\kappa(\theta) = \frac{1}{2}\triangle_\theta\hat{\Psi} = \frac{1}{2}\frac{\xi_0^2}{D_L^2}\triangle_\theta\Psi . \tag{2.24}$$

Since

$$\triangle_\theta = D_L^2\triangle_\xi = \frac{D_L^2}{\xi_0^2}\triangle_x , \tag{2.25}$$

using adimensional quantities Eq. 2.24 reads

$$\kappa(\vec{x}) = \frac{1}{2}\triangle_x\Psi(\vec{x}) \tag{2.26}$$

Integrating Eq. (2.16), the effective lensing potential can be written in terms of the convergence as

$$\Psi(\vec{x}) = \frac{1}{\pi}\int_{\mathbf{R}^2}\kappa(\vec{x}')\ln|\vec{x}-\vec{x}'|d^2x' , \tag{2.27}$$

from which we obtain that the scaled deflection angle is

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi}\int_{\mathbf{R}^2}d^2x'\kappa(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} . \tag{2.28}$$

2.3 Magnification and distortion

One of the main consequences of gravitational lensing is image distortion. This is particularly evident when the source has an extended size. For example, background galaxies can appear as very long arcs when lensed by galaxy clusters or other galaxies.

The distortion arises because light bundles are deflected differentially. Ideally, the shape of the images can be determined by solving the lens equation for all the points within the extended source. In particular, if the source is much smaller than the angular scale on which the lens deflection angle field change, the relation between source and image positions can locally be linearized. In other words, the distortion of images can be described by the Jacobian matrix

$$A \equiv \frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \left(\delta_{ij} - \frac{\partial\alpha_i(\vec{\theta})}{\partial\theta_j}\right) = \left(\delta_{ij} - \frac{\partial^2\hat{\Psi}(\vec{\theta})}{\partial\theta_i\partial\theta_j}\right) , \tag{2.29}$$