GRAVITATIONAL LENSING

LESSON 1 – DEFLECTION OF LIGHT

Massimo Meneghetti
AA 2016-2017

TEACHER

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ricevimento: da concordare via e-mail o telefono

google group:

https://groups.google.com/d/forum/gravlens_2017

THE COURSE

- module 1: Basics of Gravitational Lensing Theory
- Applications of Gravitational Lensing:
  - module 2: microlensing in the MW
  - module 3: lensing by galaxies
  - module 4: lensing by galaxy clusters
  - module 5: lensing by the LSS
- Python
- Final exam

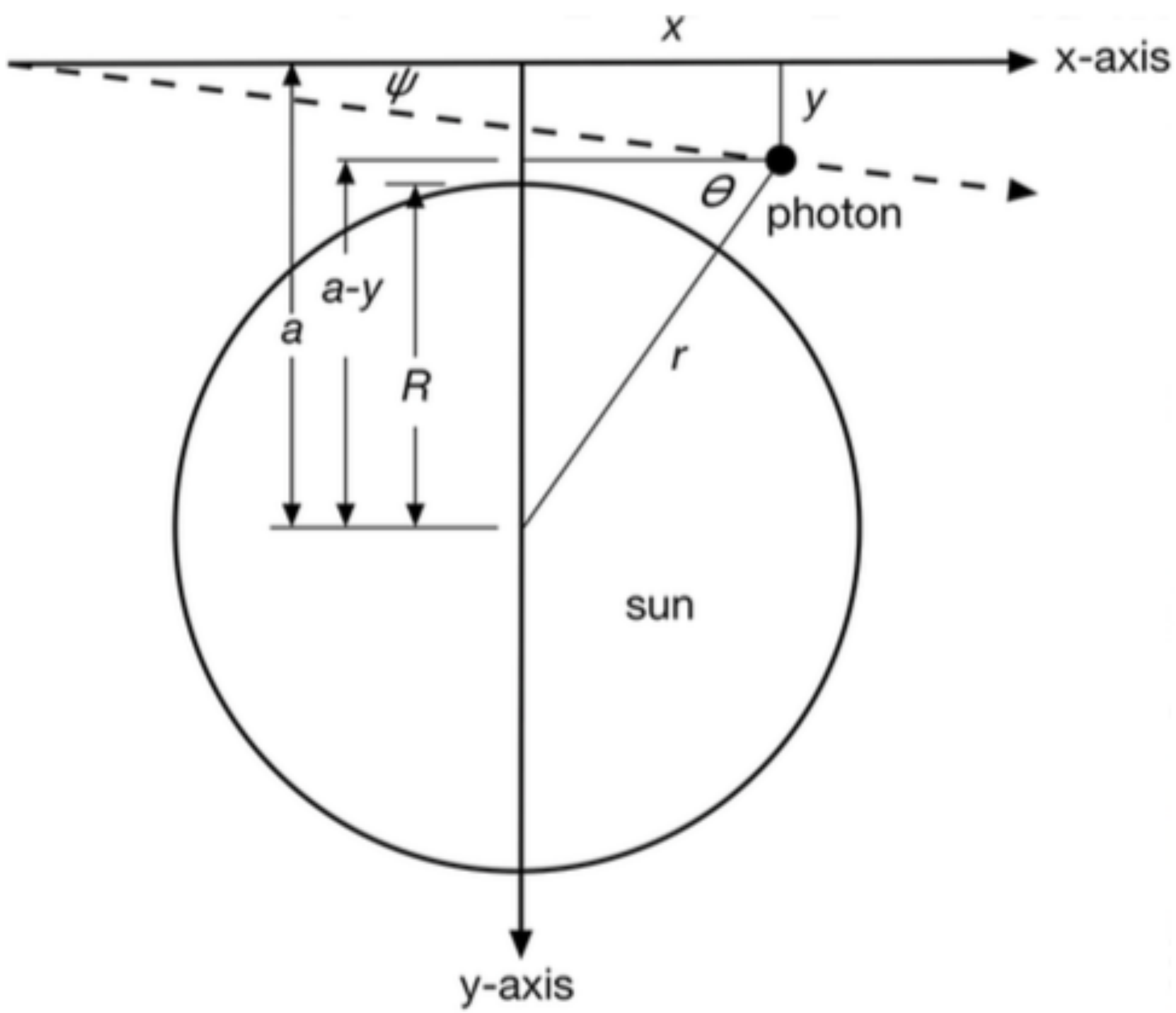
LEARNING RESOURCES

- http://pico.bo.astro.it/~massimo/teaching.html
- available materials:
  - lecture scripts, articles, tutorials, links to external material and books
  - slides
  - python notebooks

CONTENTS OF TODAY'S LESSON

- Deflection of light in the Newtonian limit
- Gravitational lensing in the context of general relativity
- The deflection angle

DEFLECTION OF A LIGHT CORPUSCLE



- Assumptions:
  - photons have an inertial gravitational mass
  - photons propagate at speed of light
  - Newton's law of gravity
  - Newton's principle of equivalence

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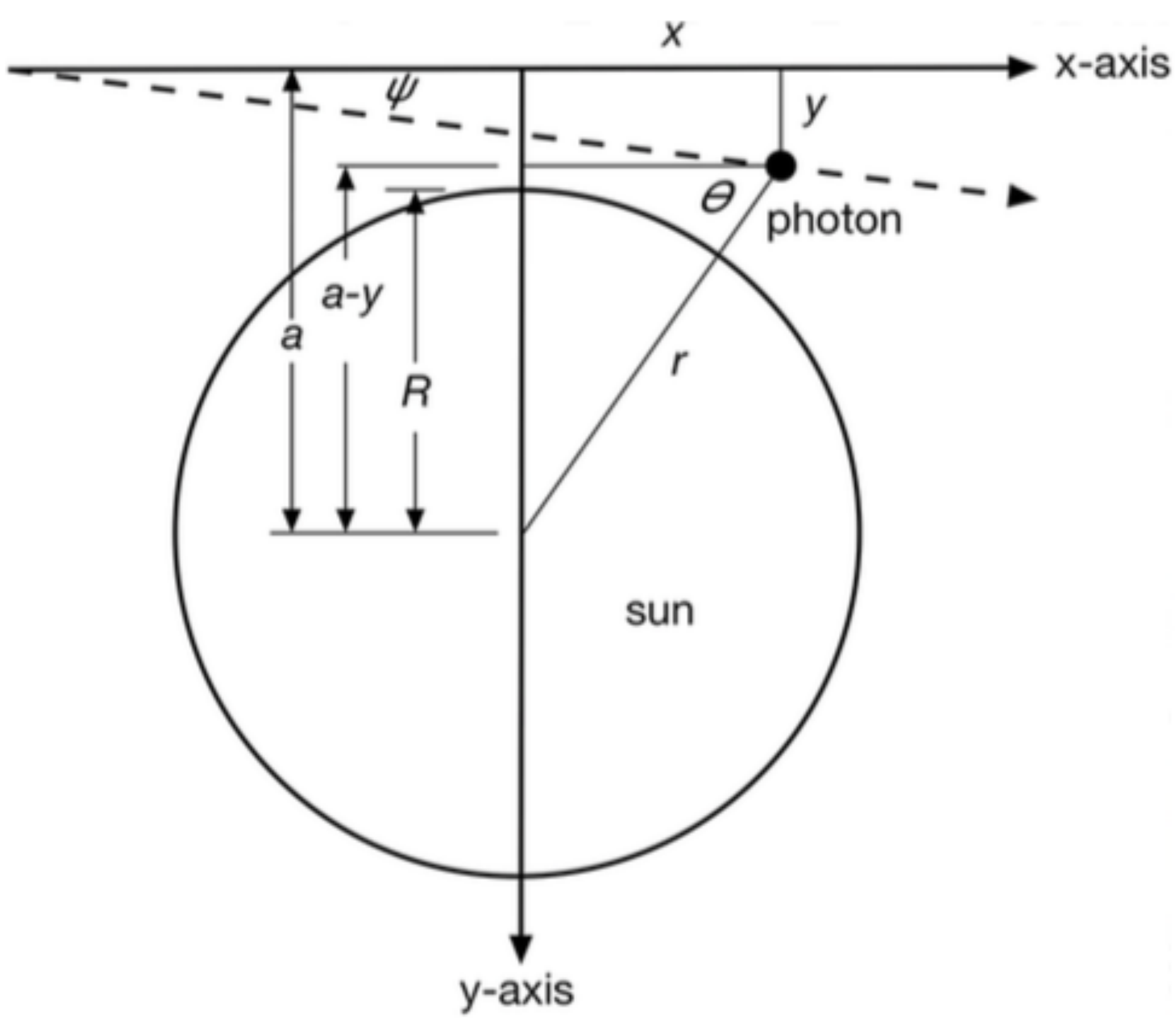

$$m = \frac{p}{c}$$

$$x = ct$$

$$\begin{aligned}\vec{F} &= \frac{d\vec{p}}{dt} \\ &= |F|(\cos\theta, \sin\theta) \\ &= \frac{GMm}{r^2}(\cos\theta, \sin\theta)\end{aligned}$$

$$r^2 = x^2 + (a-y)^2$$

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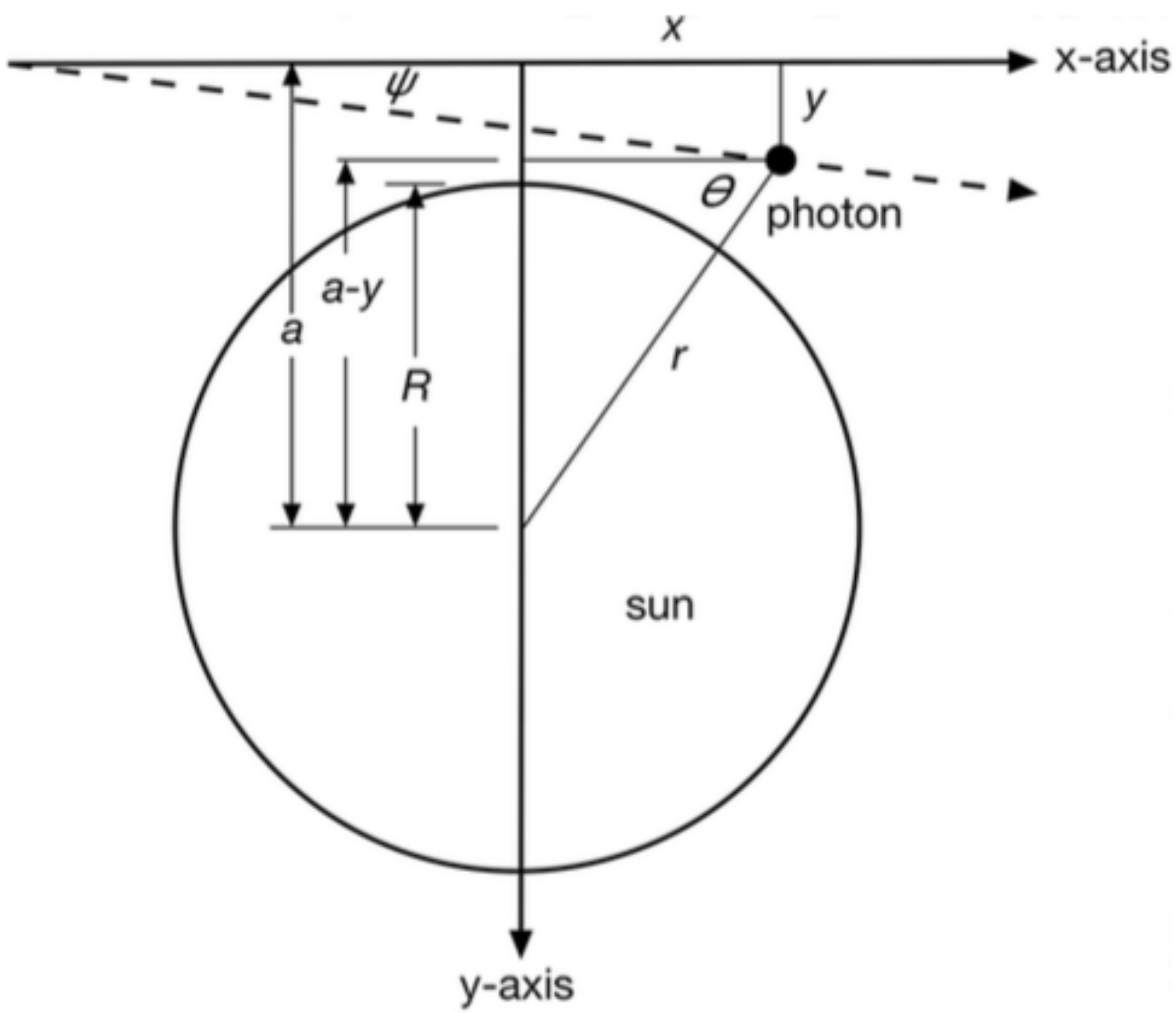

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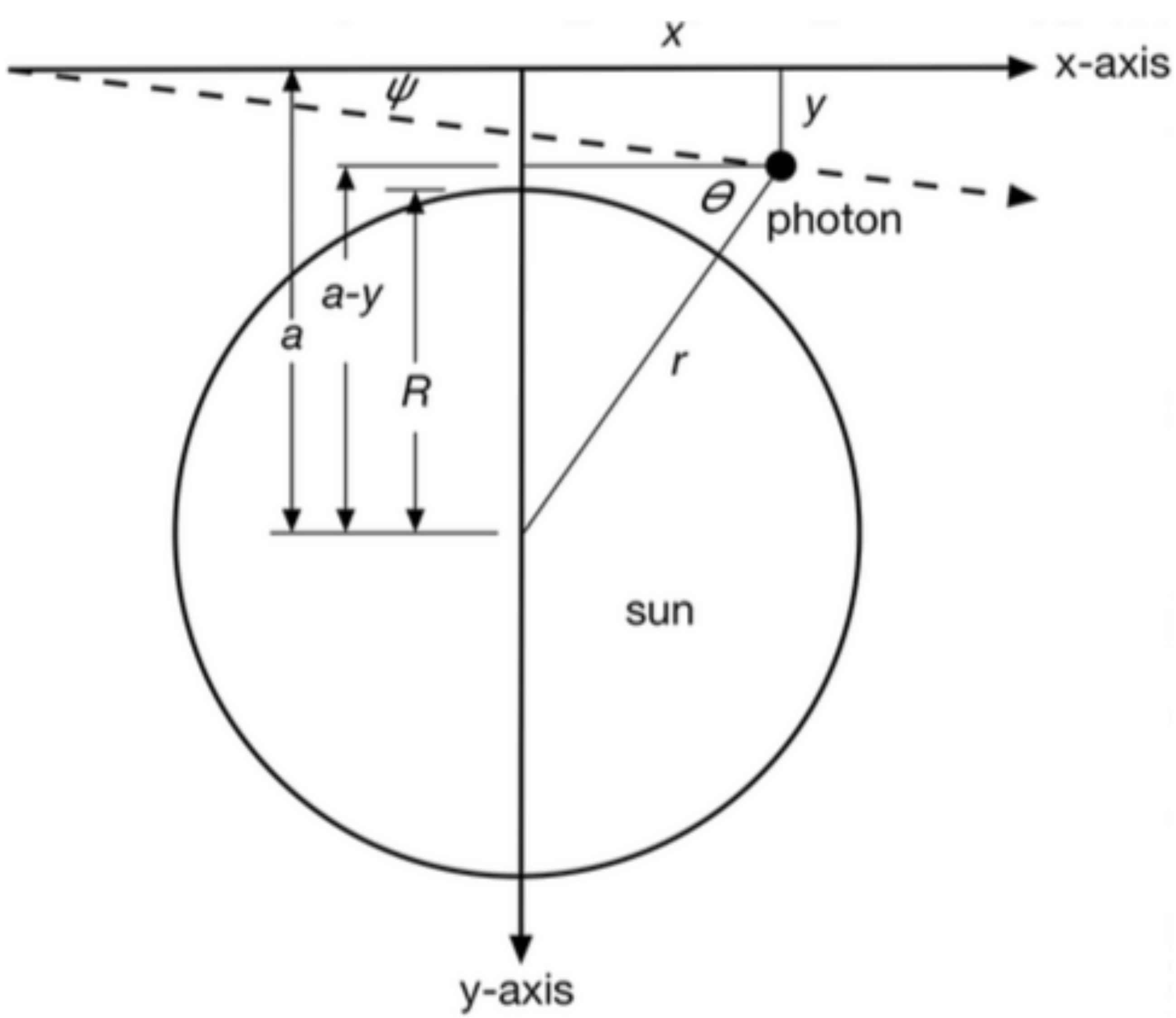

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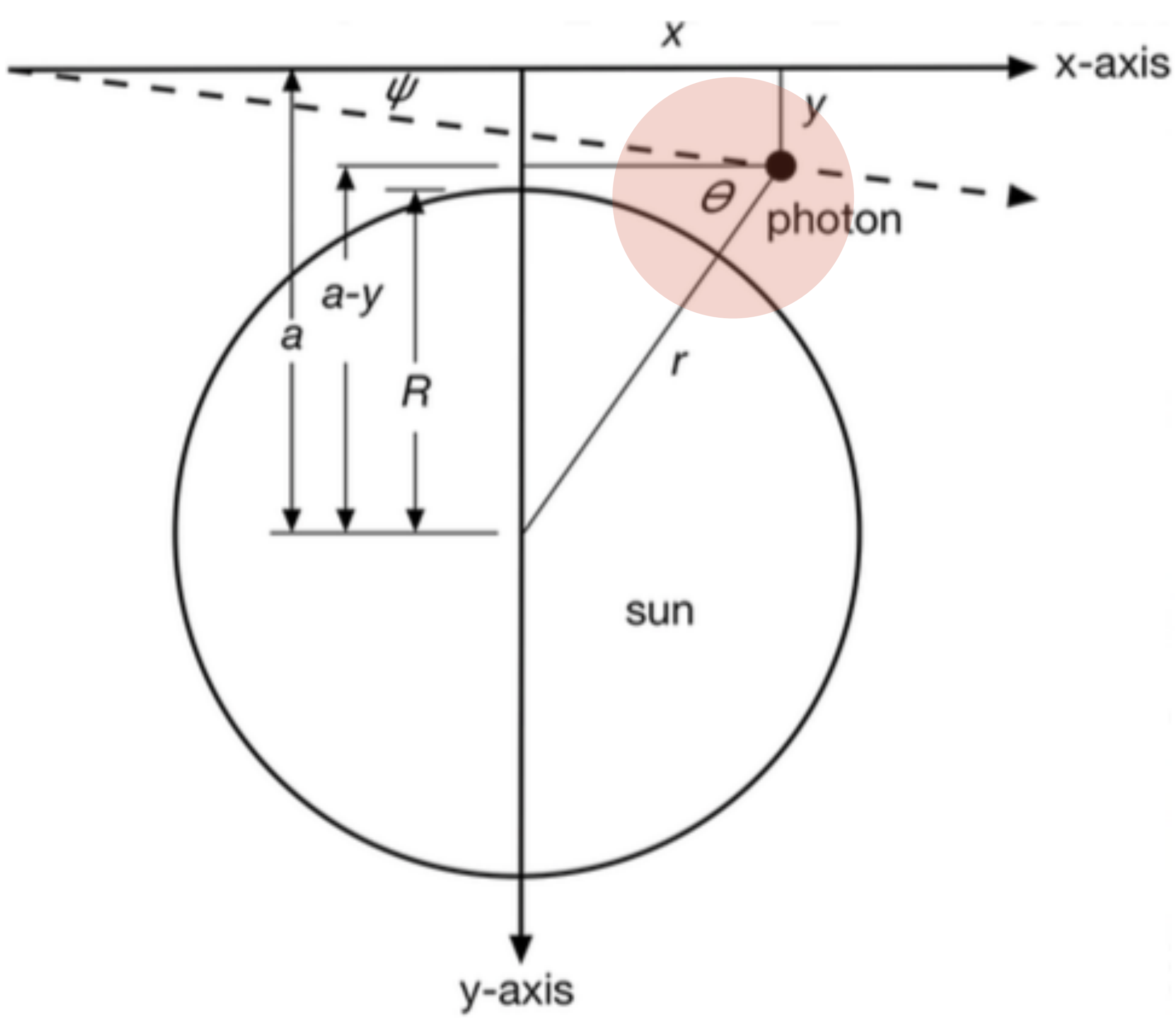

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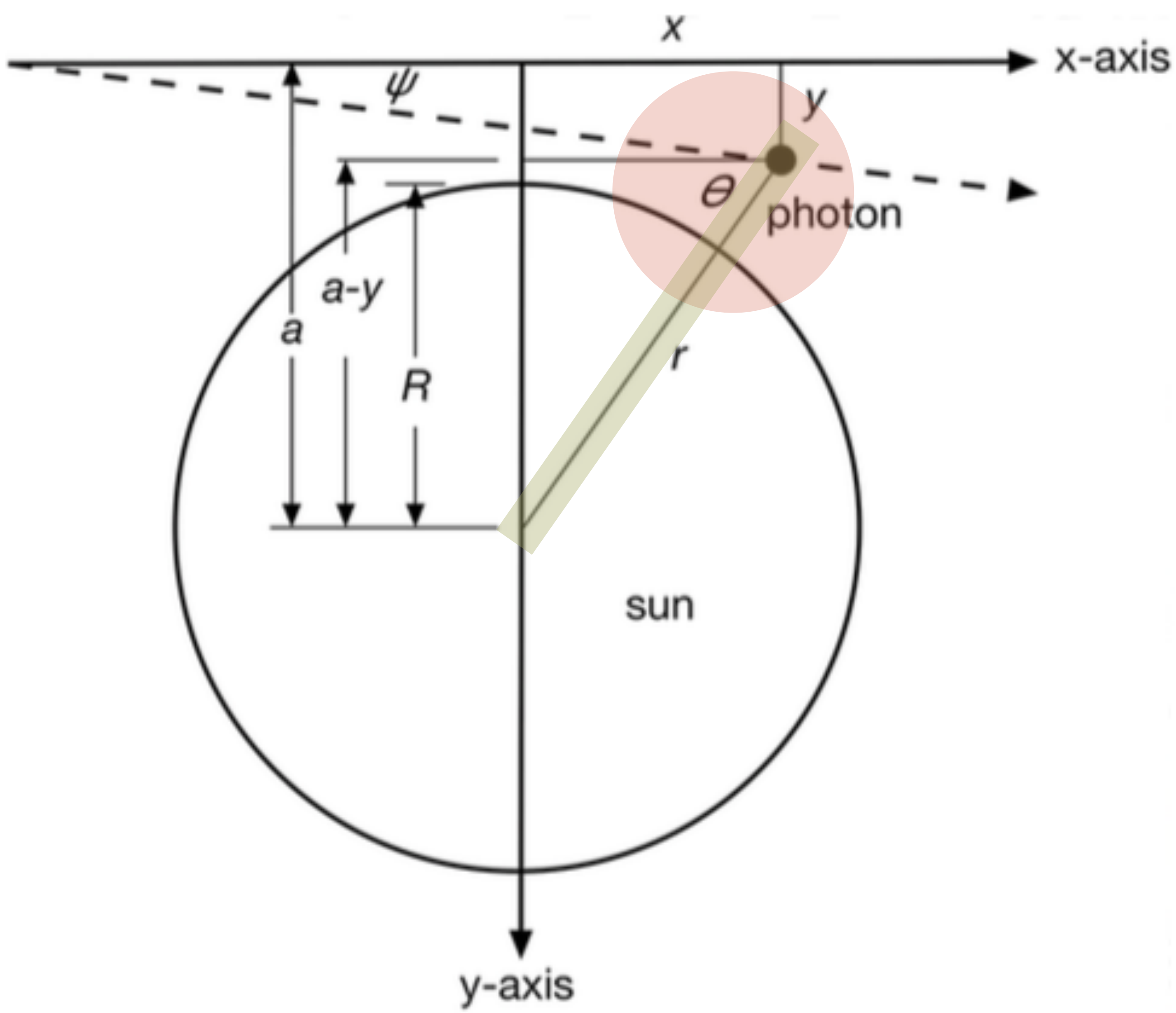

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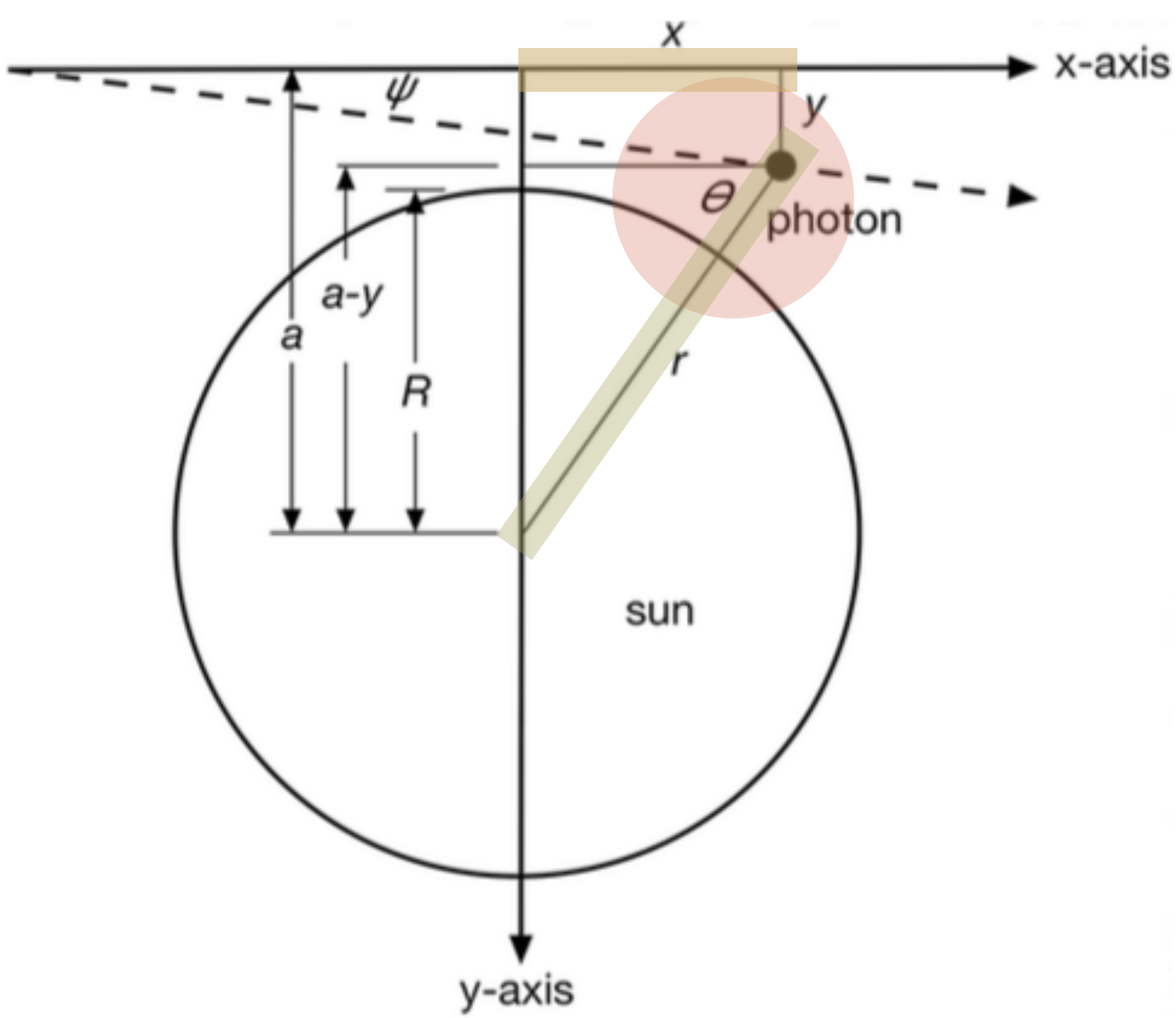

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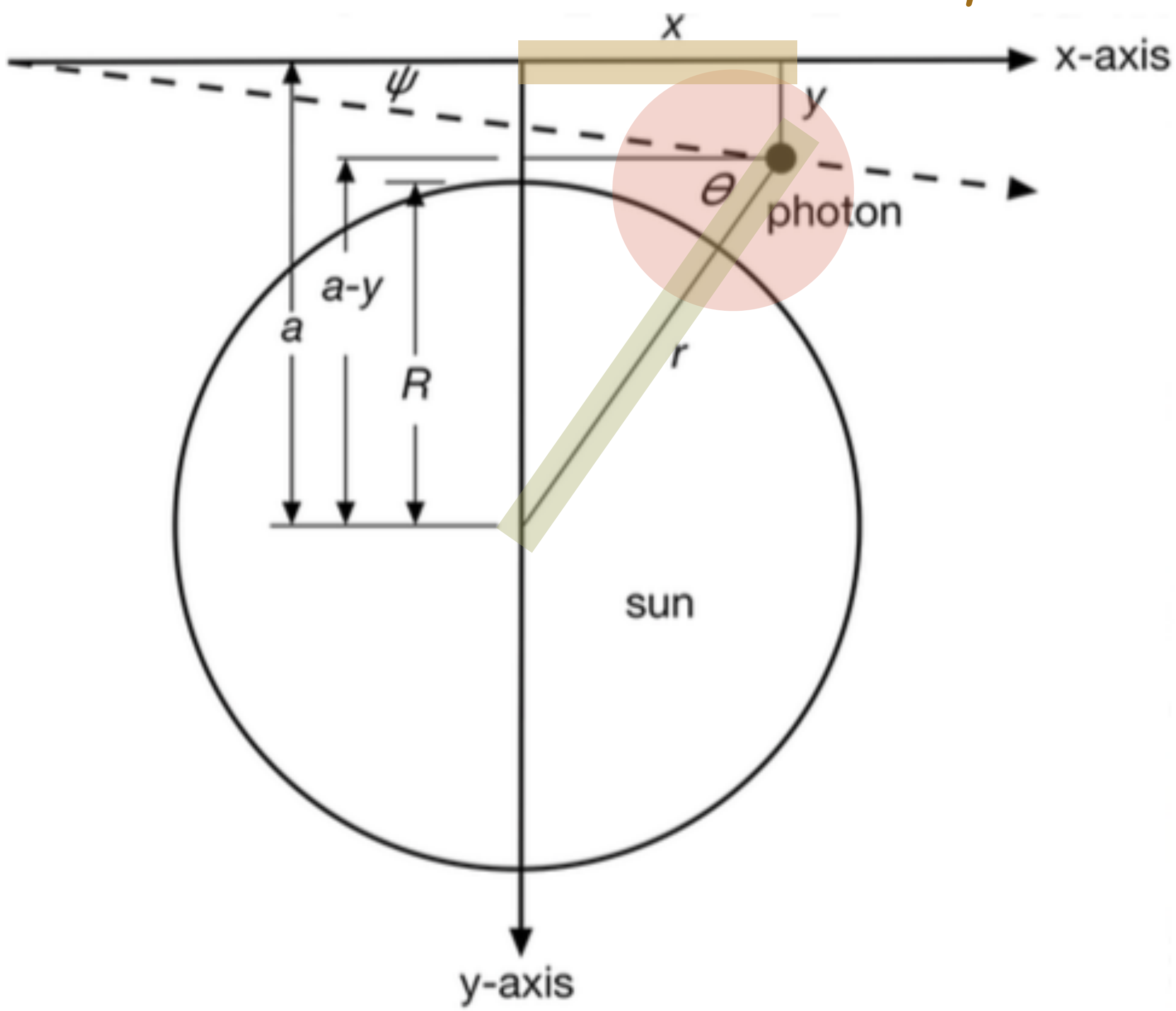
DEFLECTION OF A LIGHT CORPUSCLE

$$\cos\theta = \frac{x}{r}$$



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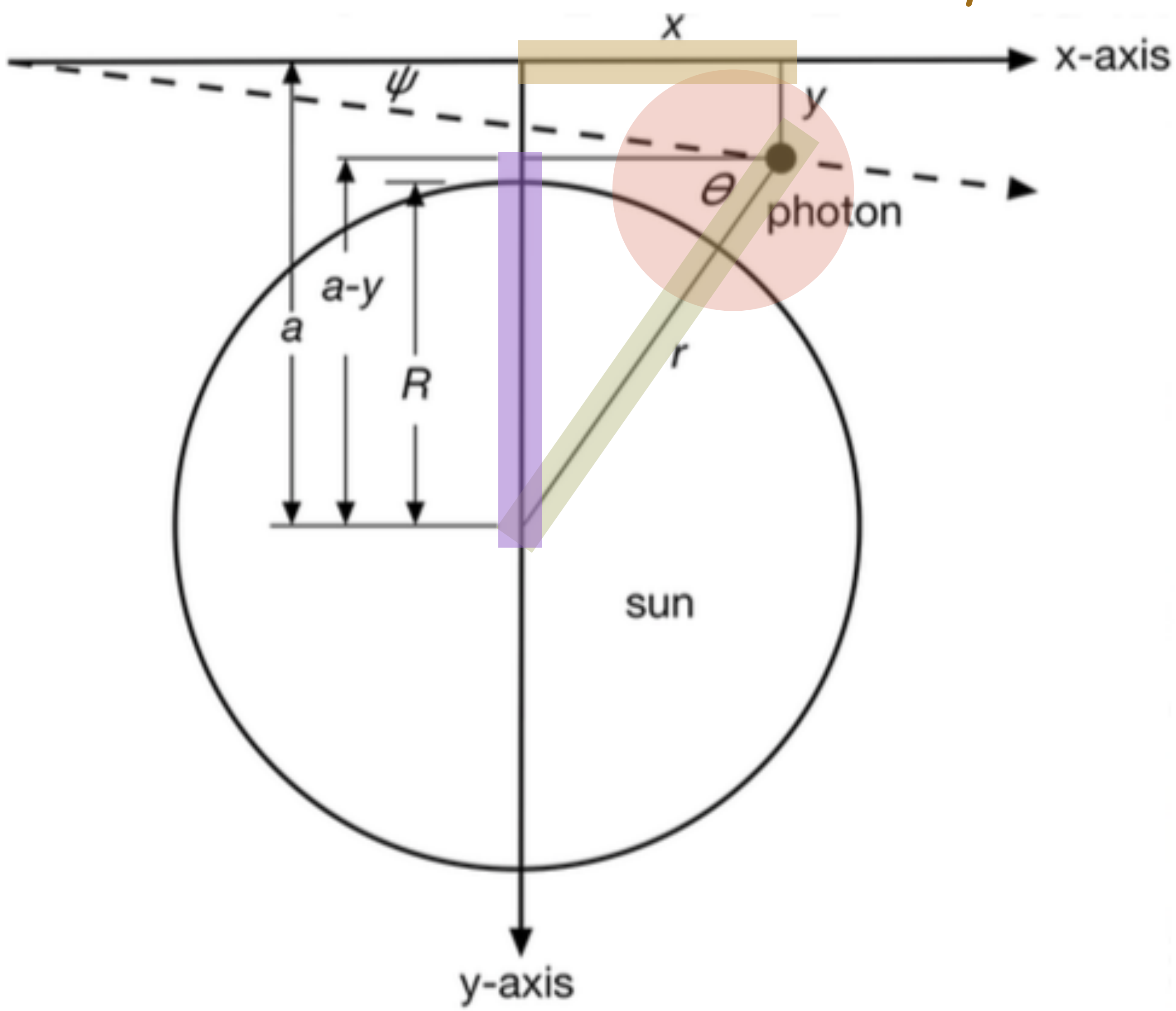

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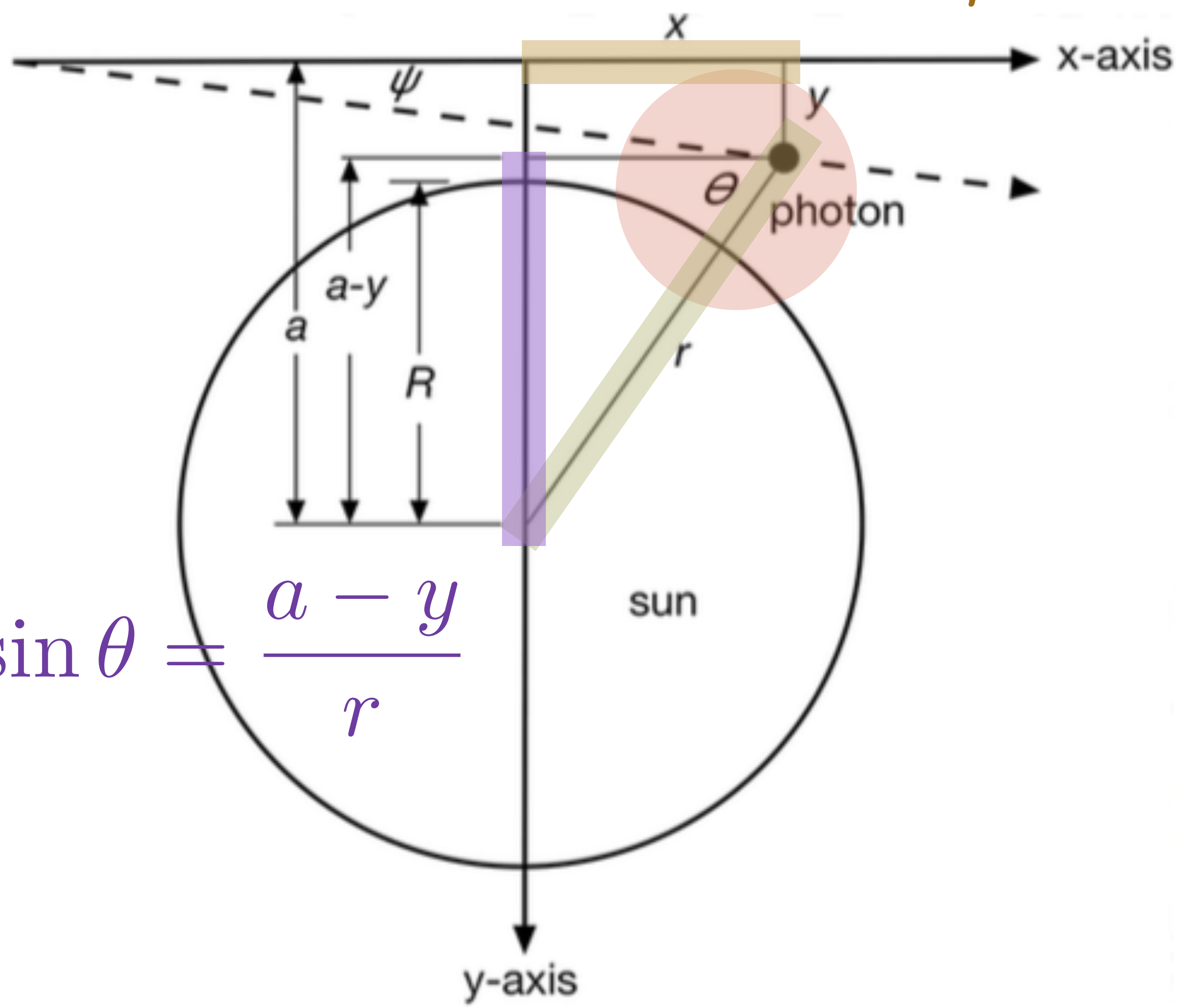

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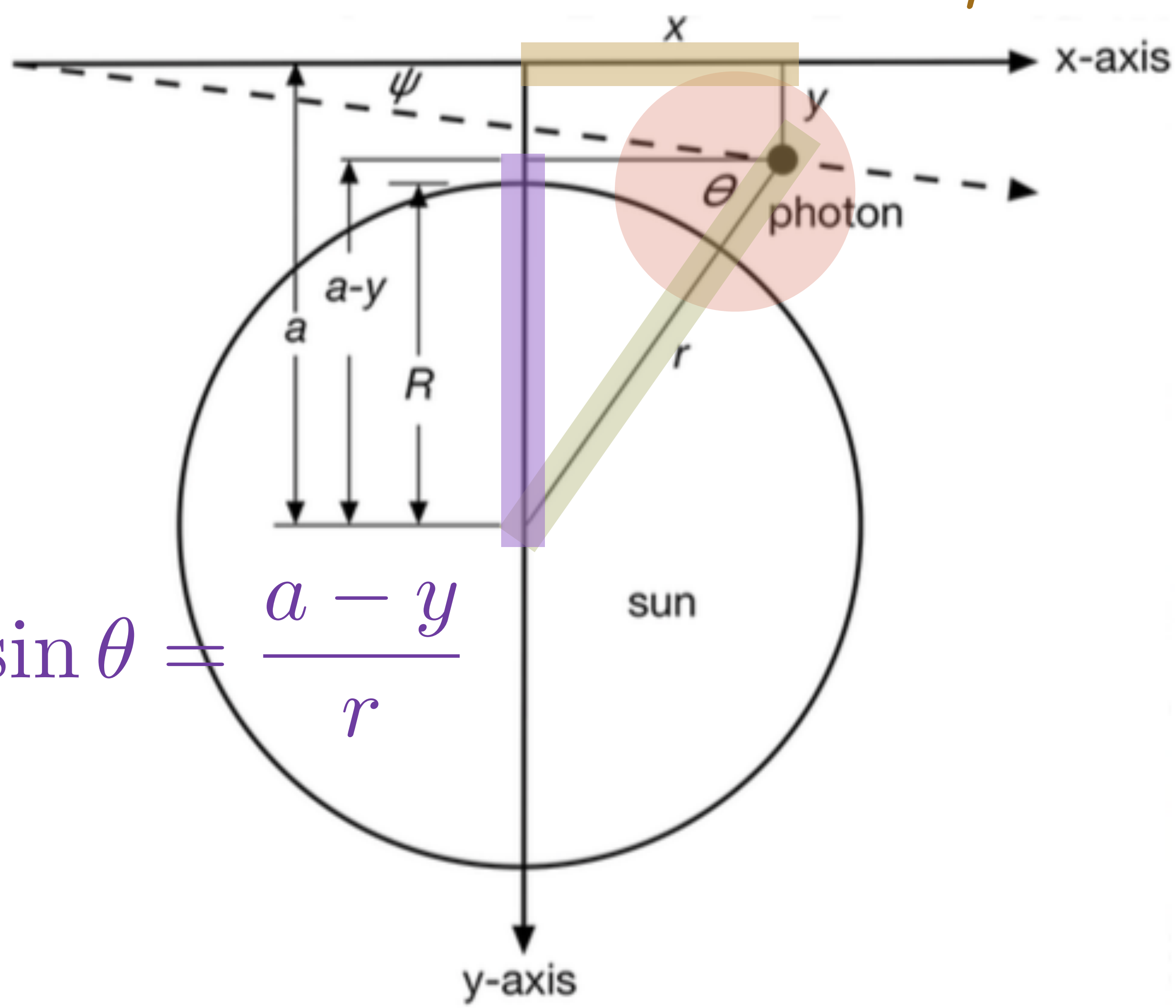

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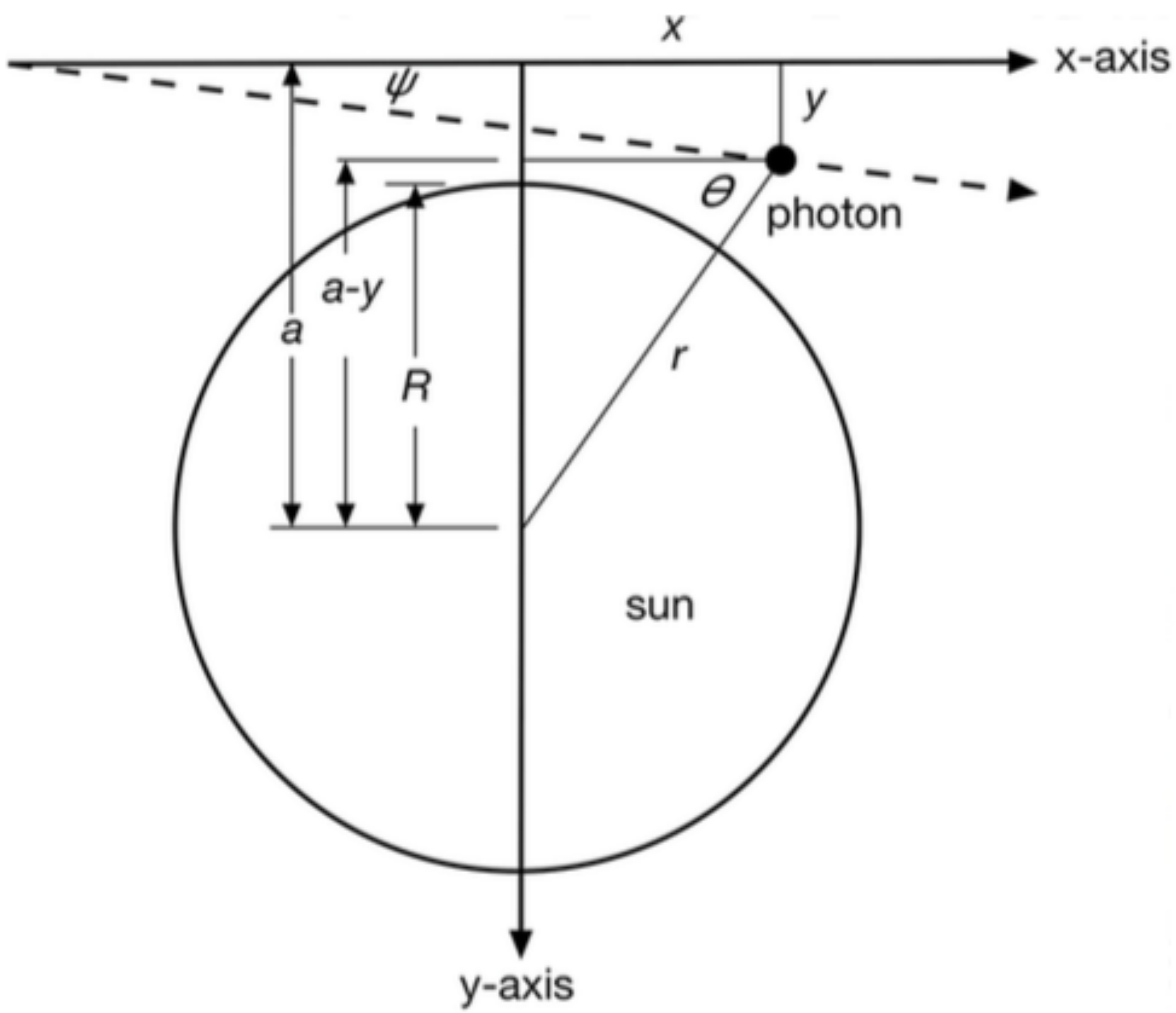

$$x = ct$$

$$dx = cdt$$

$$\frac{dp_i}{dt} = \frac{dp_i}{dx}\frac{dx}{dt} = c\frac{dp_i}{dx}$$

$$\frac{dp_x}{dx} = \frac{GMp}{c^2}\frac{x}{(x^2+(a-y)^2)^{3/2}}$$

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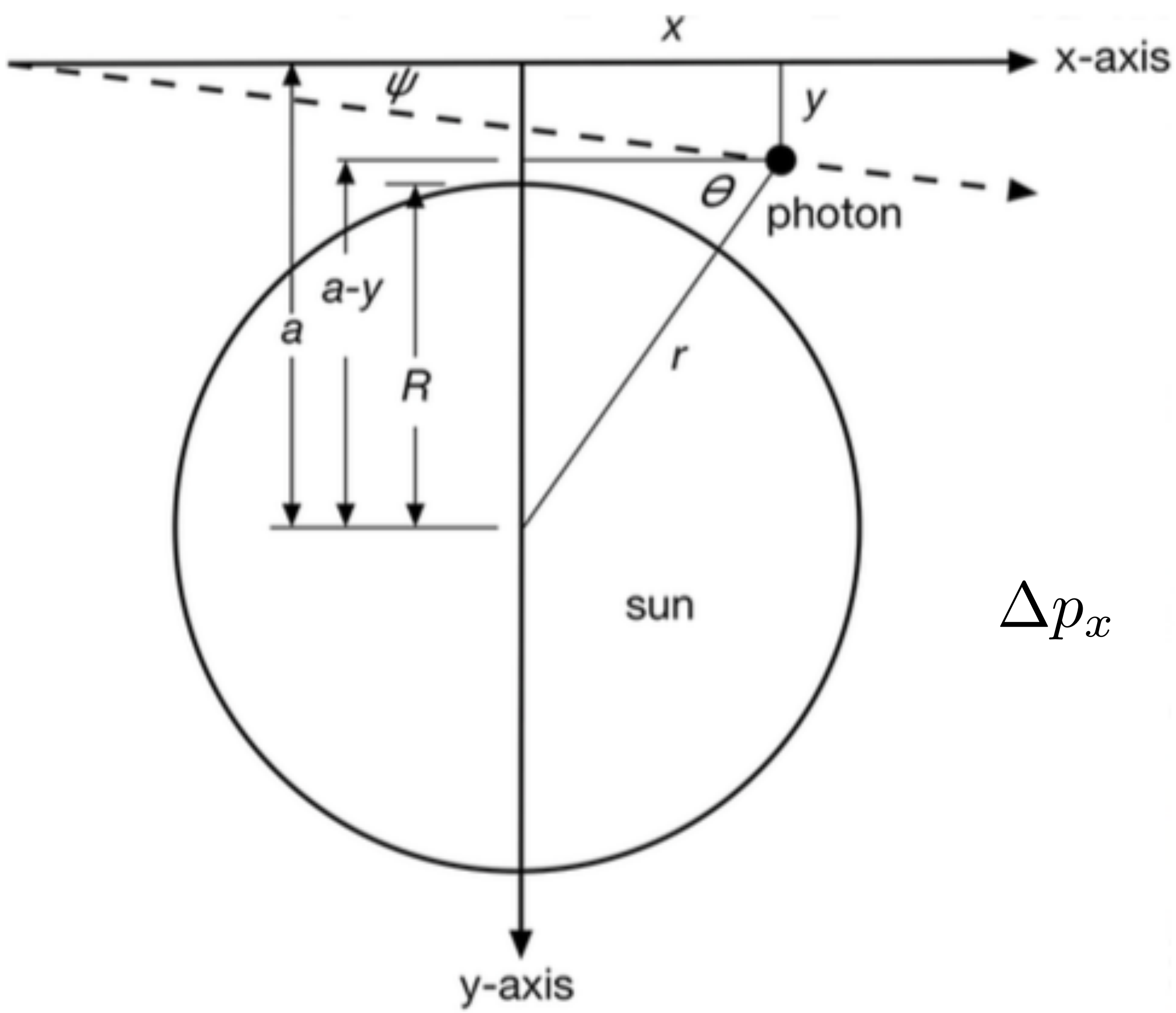

$$\frac{dp_x}{dx} = \frac{GMp}{c^2}\frac{x}{(x^2 + (a-y)^2)^{3/2}}$$

$$\begin{aligned}\Delta p_x &= \frac{GMp}{c^2}\int_{-\infty}^{+\infty}\frac{x}{(x + (a-y)^2)^{3/2}}dx \\ &= \frac{GMp}{c^2}\left[\log[(a-y)^2 + x^2]\right]_{-\infty}^{+\infty} \\ &= 0\end{aligned}$$

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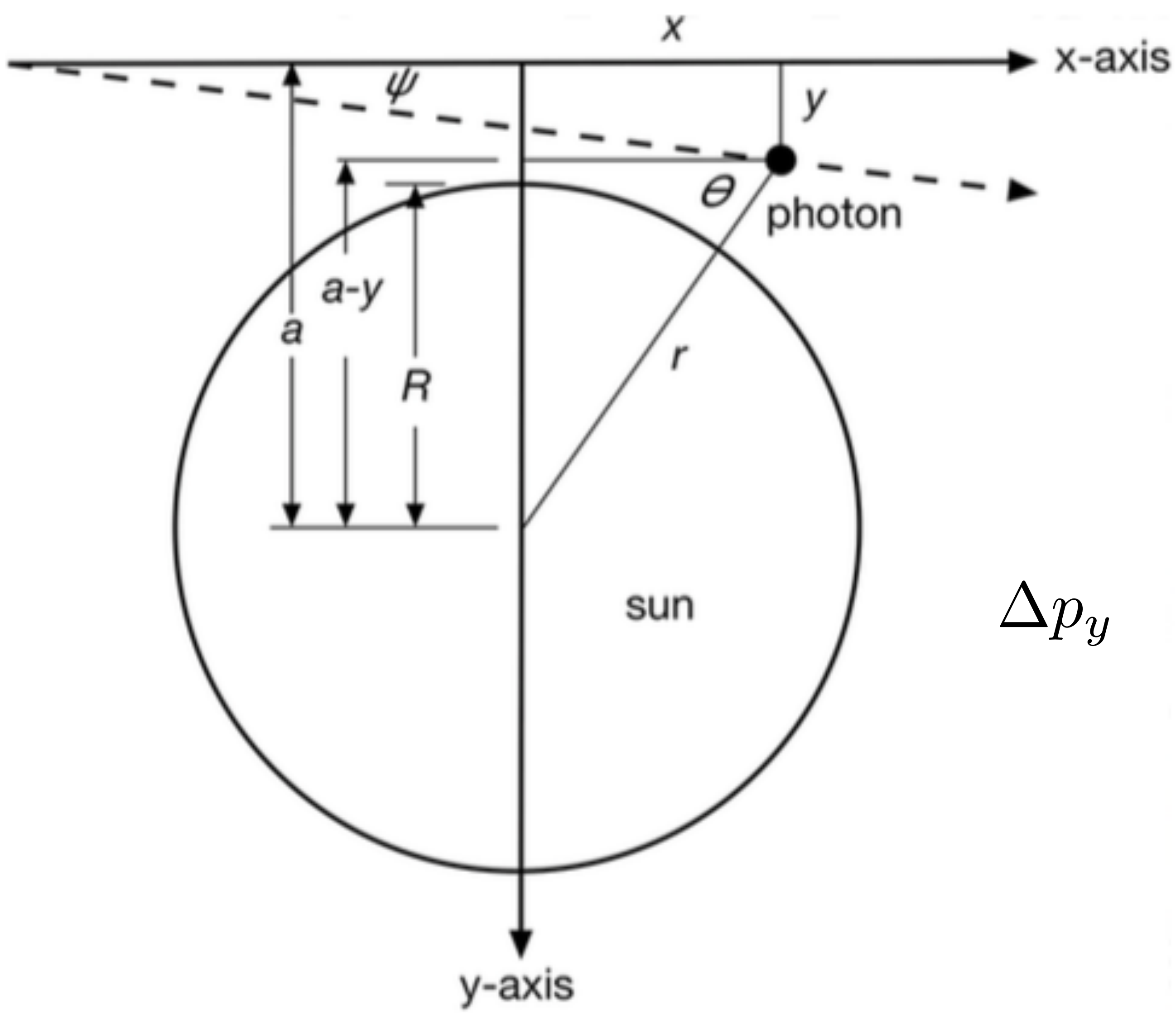

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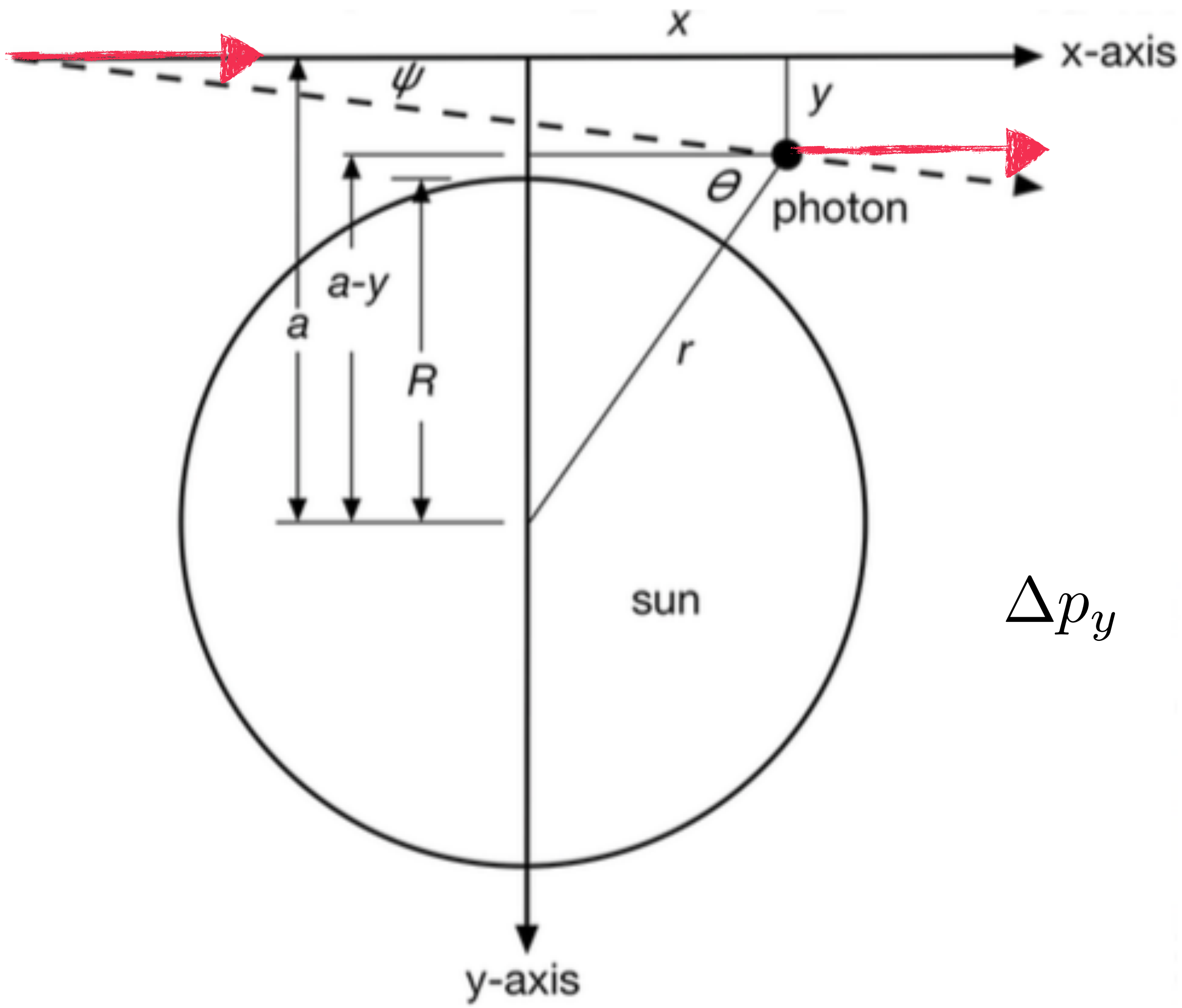

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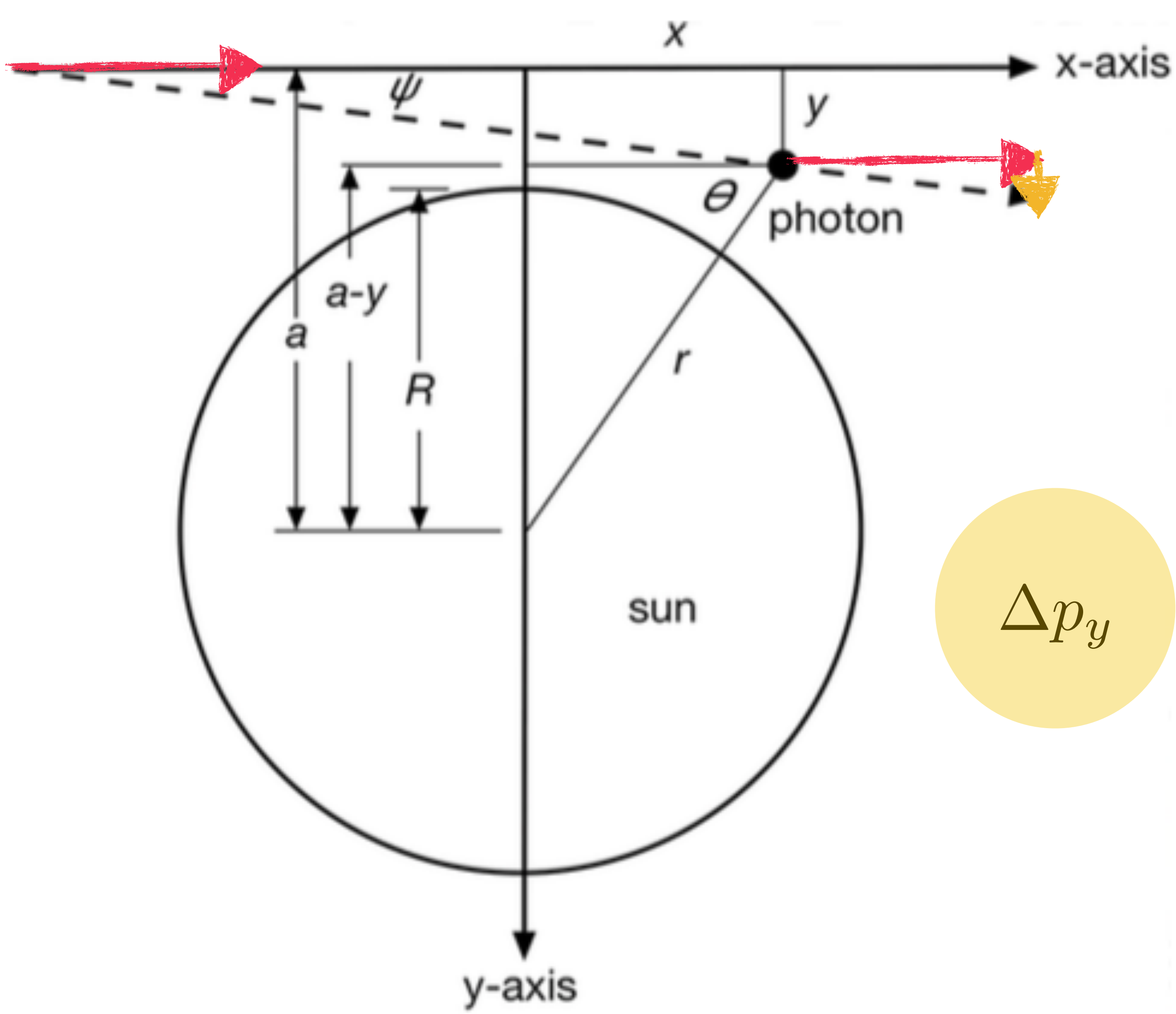

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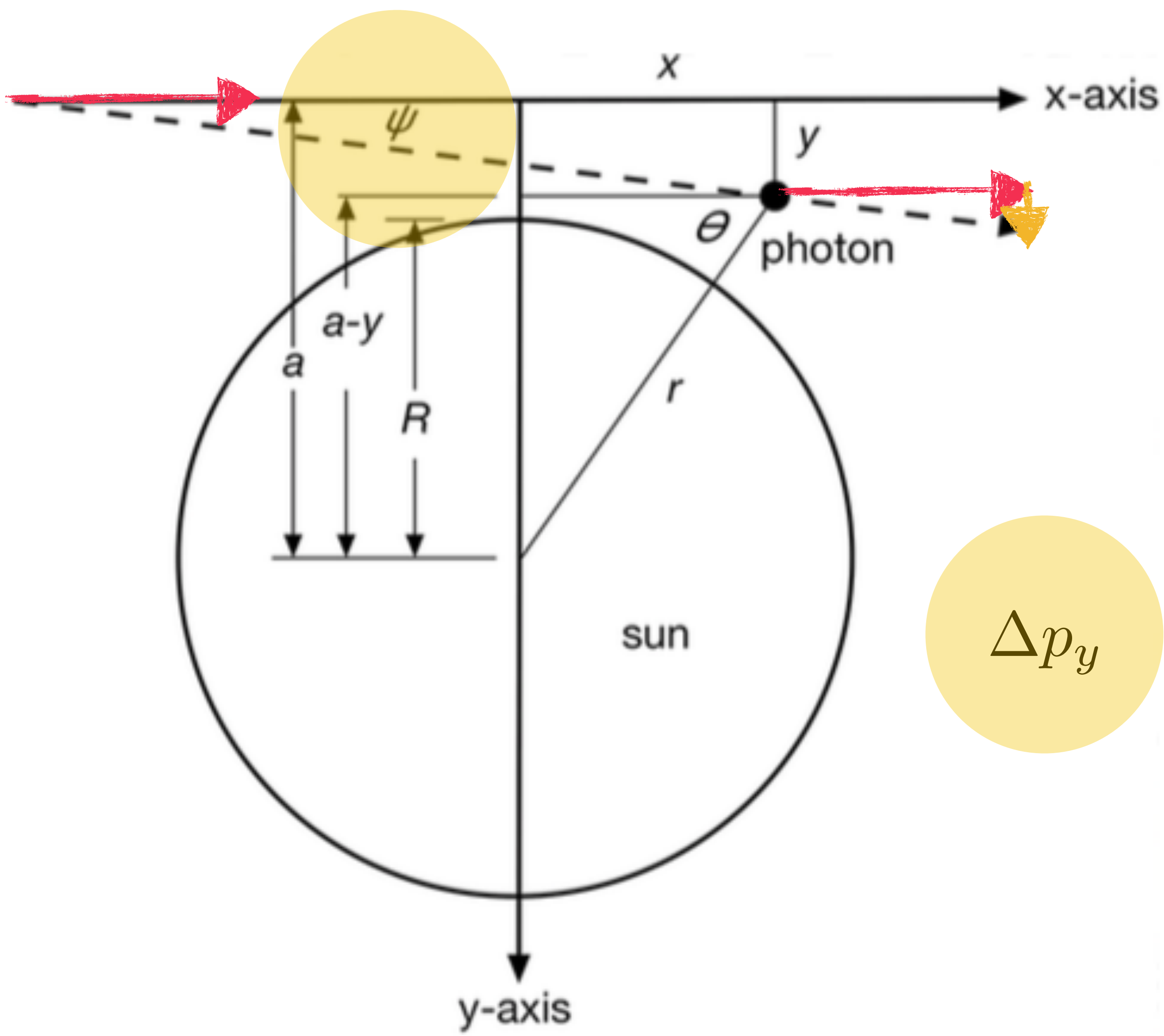

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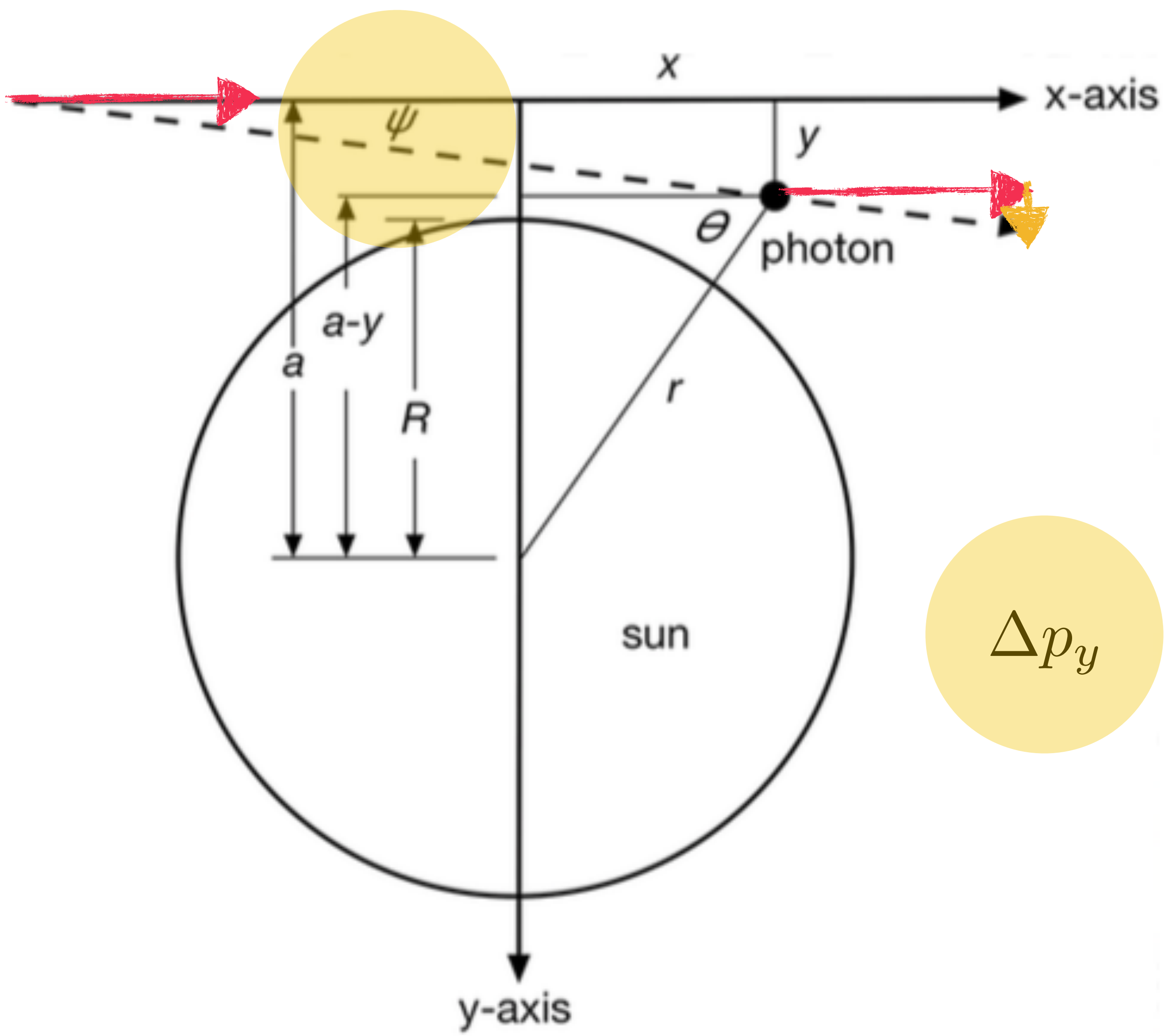

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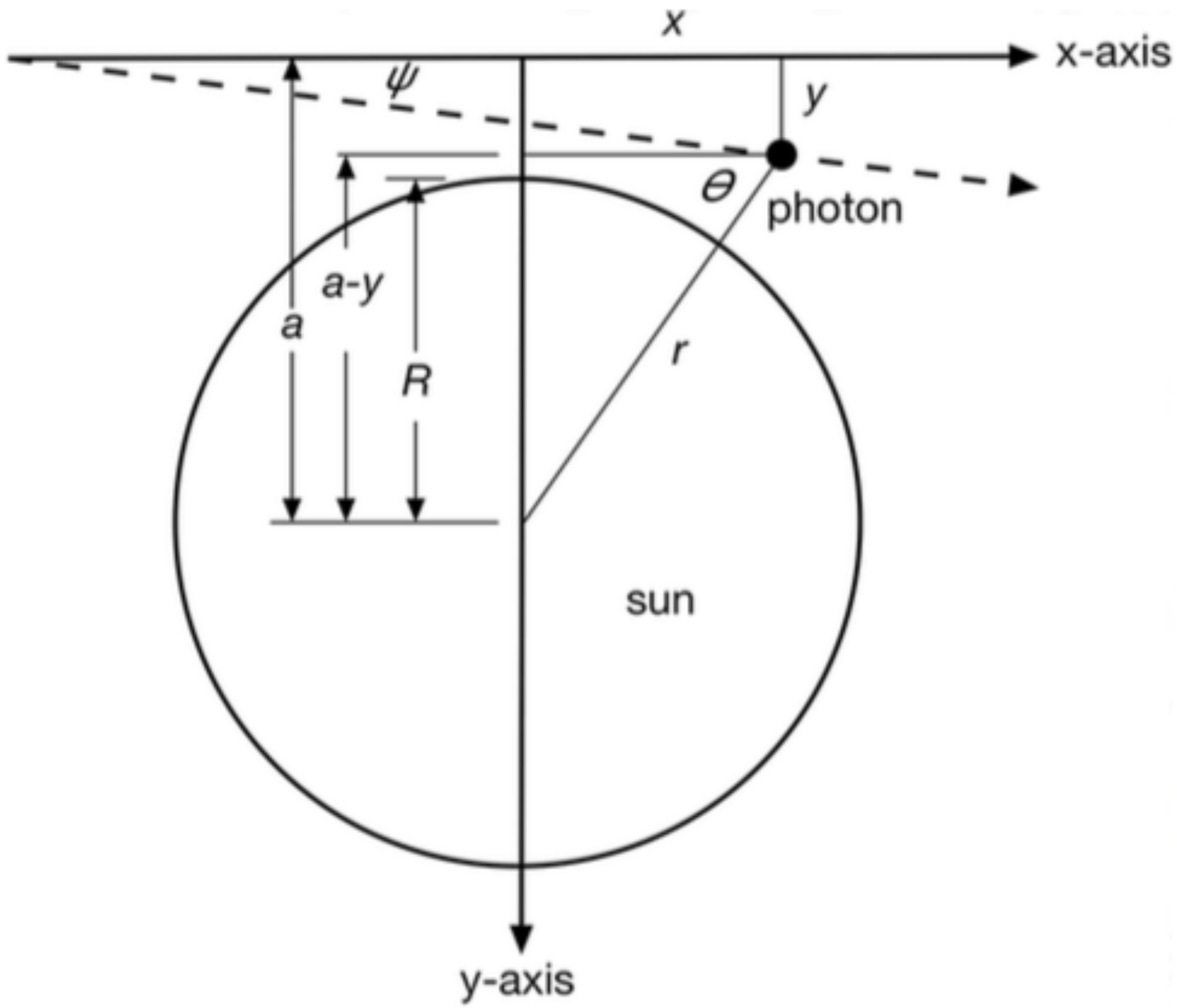
$$\psi = \frac{\Delta p_y}{p} = \frac{2GM}{c^2}\frac{1}{a-y}$$

DEFLECTION OF A LIGHT CORPUSCLE BY THE SUN



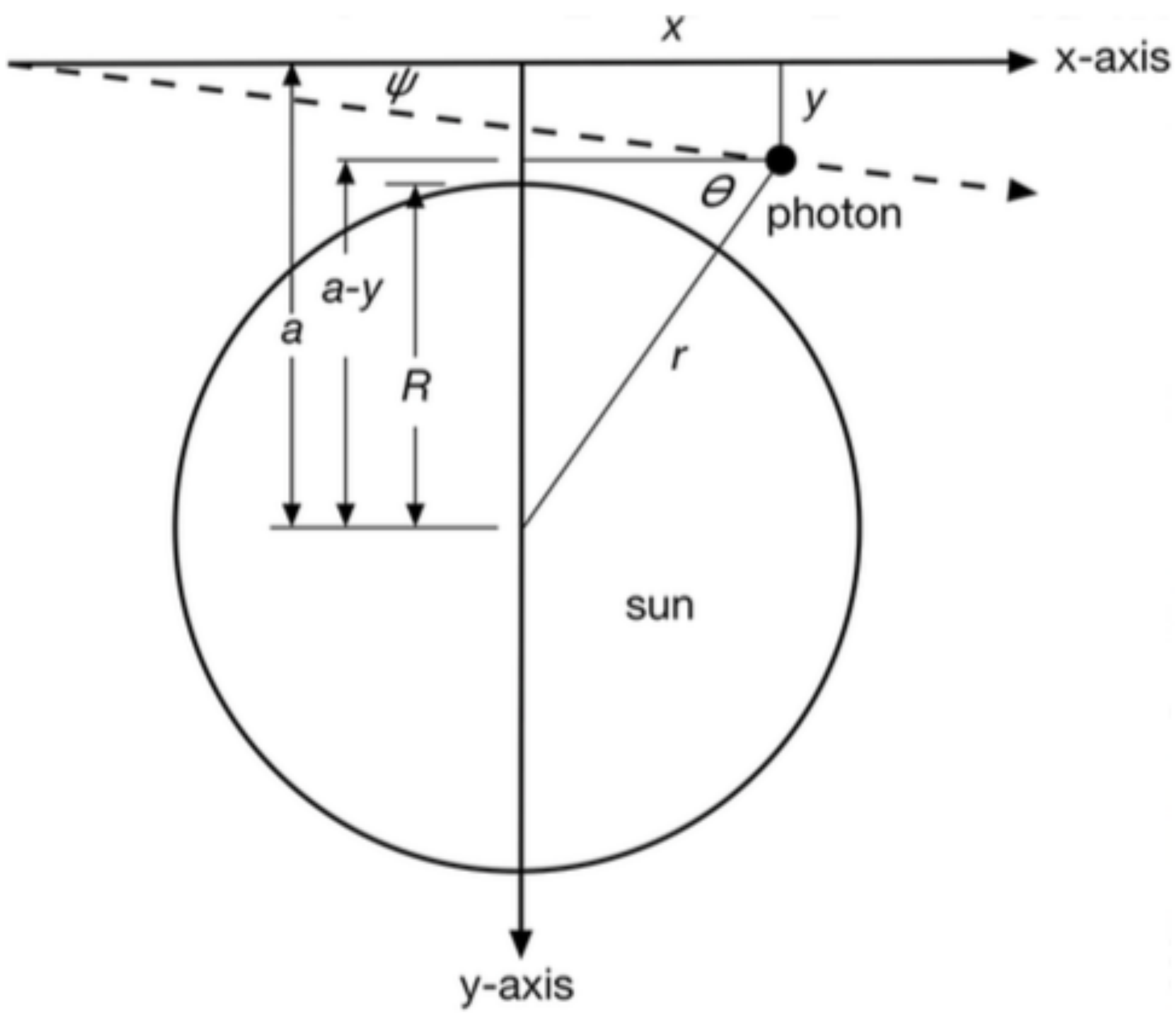
$$a - y = R_{\odot}$$

$$M = M_{\odot} = 1.989 \times 10^{30} kg$$

$$a - y = R_{\odot} = 6.96 \times 10^{8} m$$

$$\psi \approx 0.875''$$

DEFLECTION OF A LIGHT CORPUSCLE BY THE SUN



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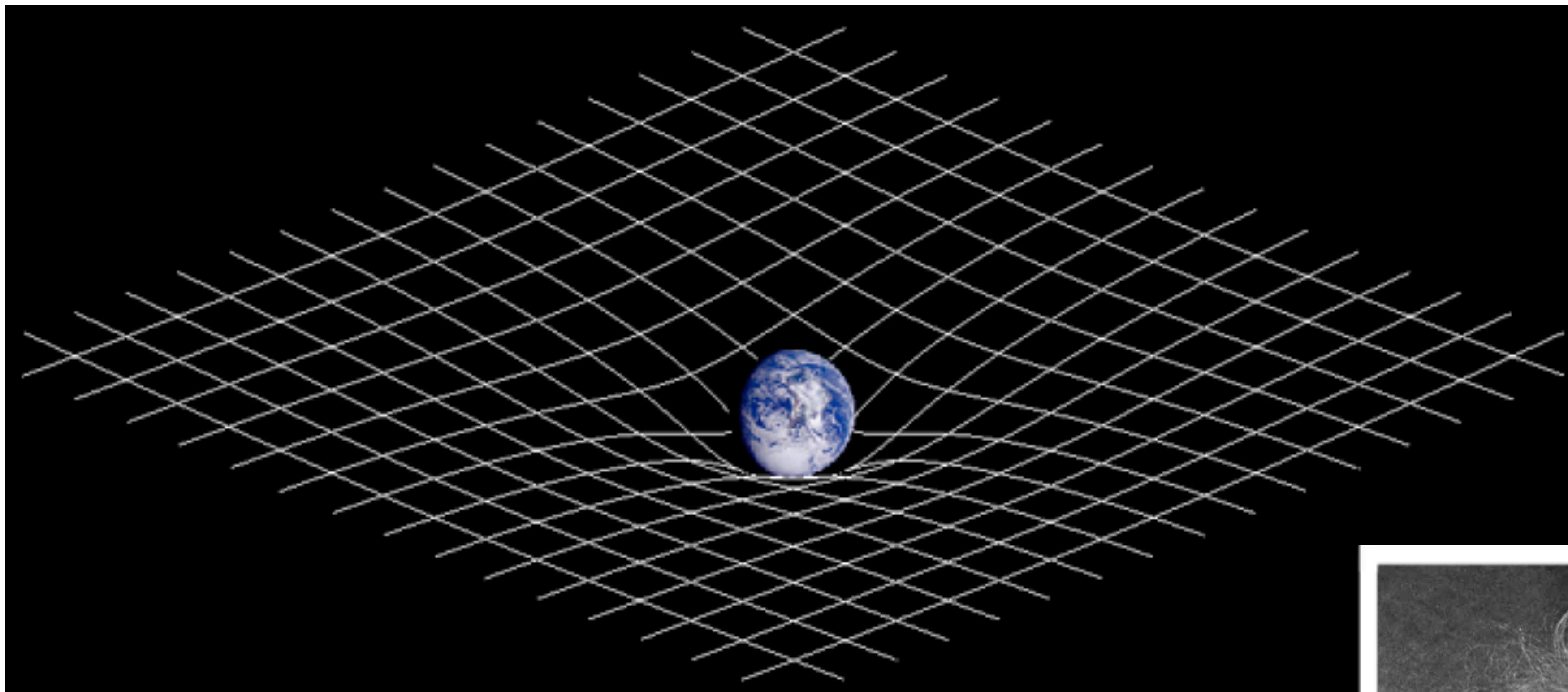
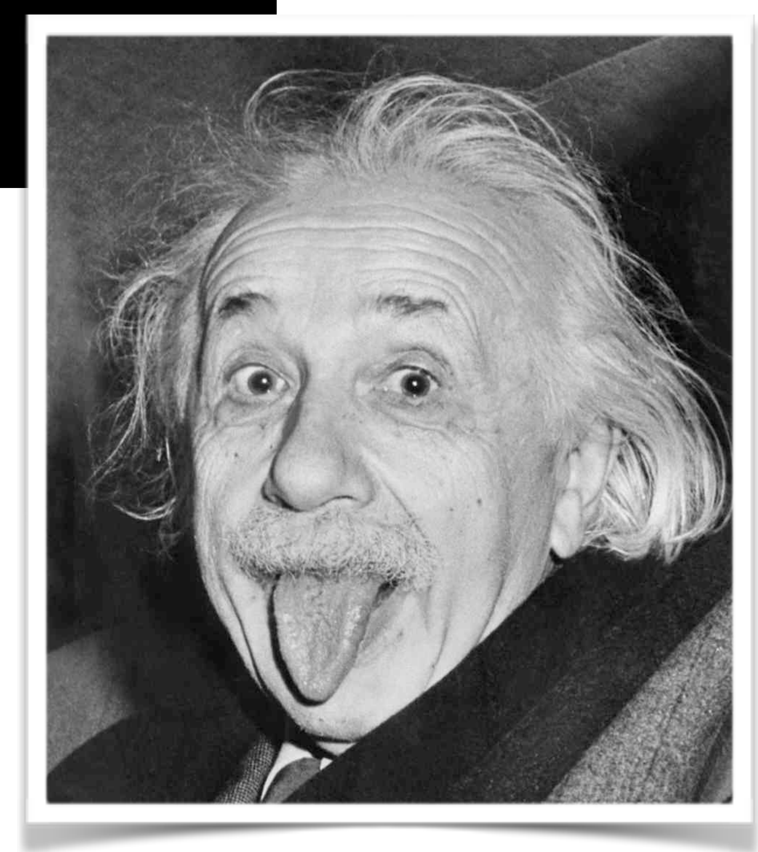
DEFLECTION OF LIGHT IN GENERAL RELATIVITY





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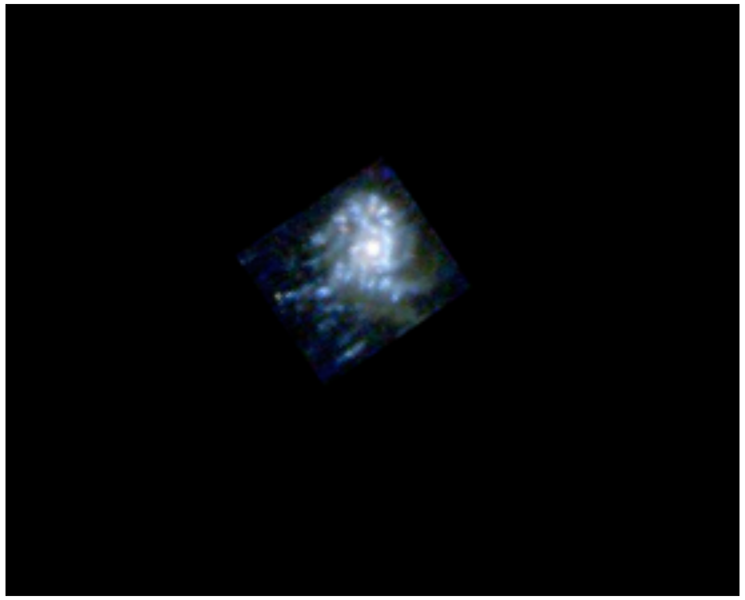
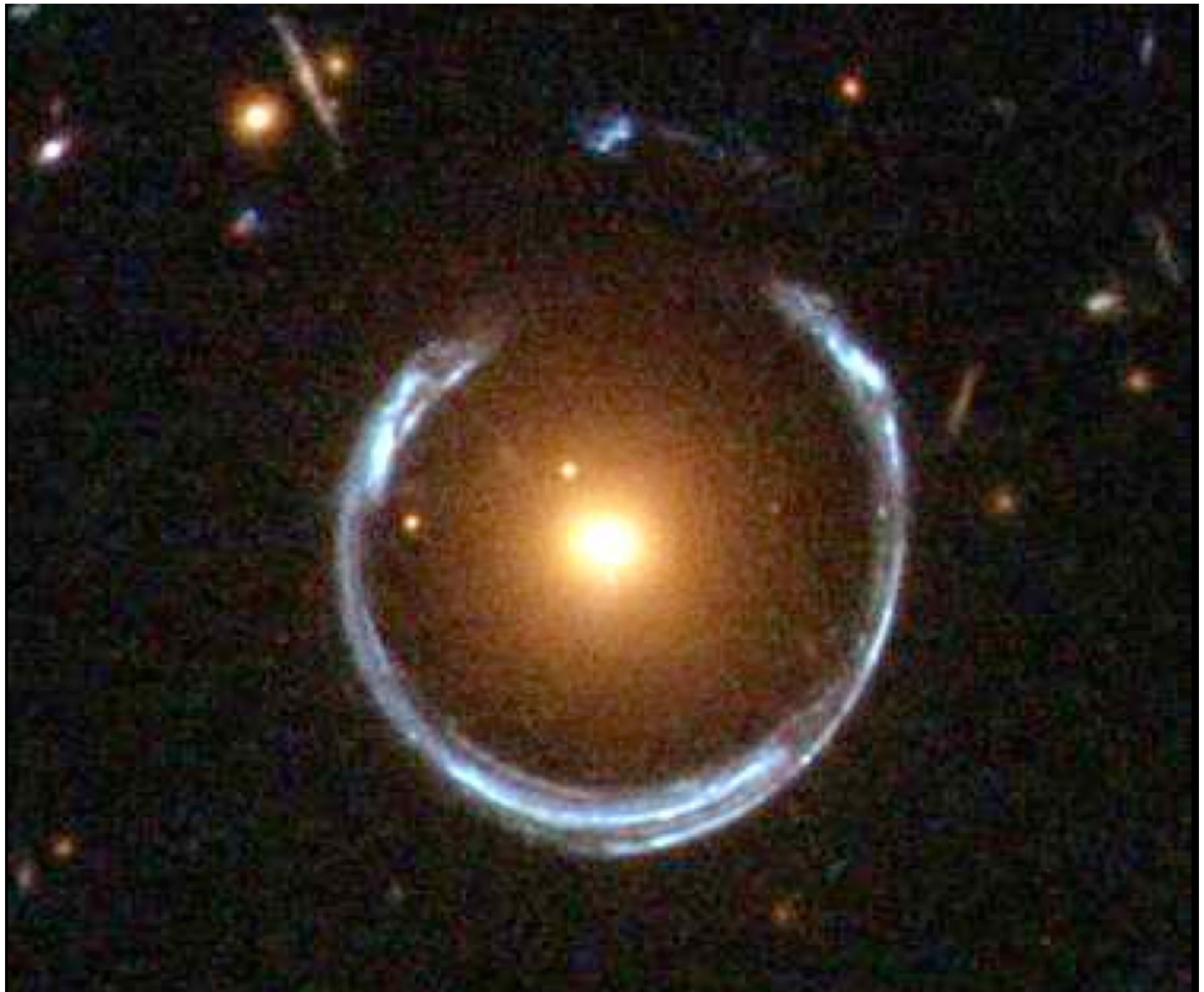
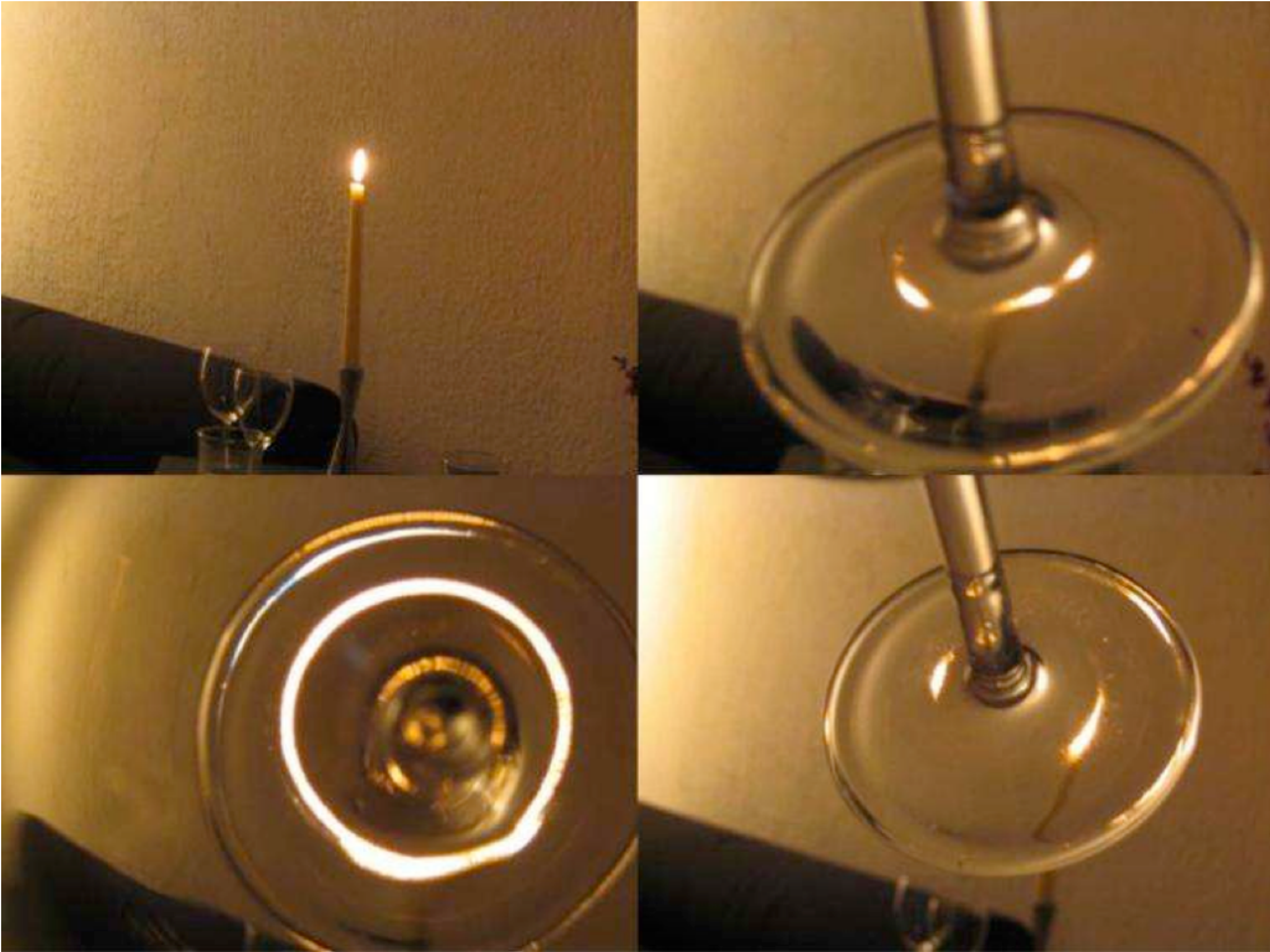


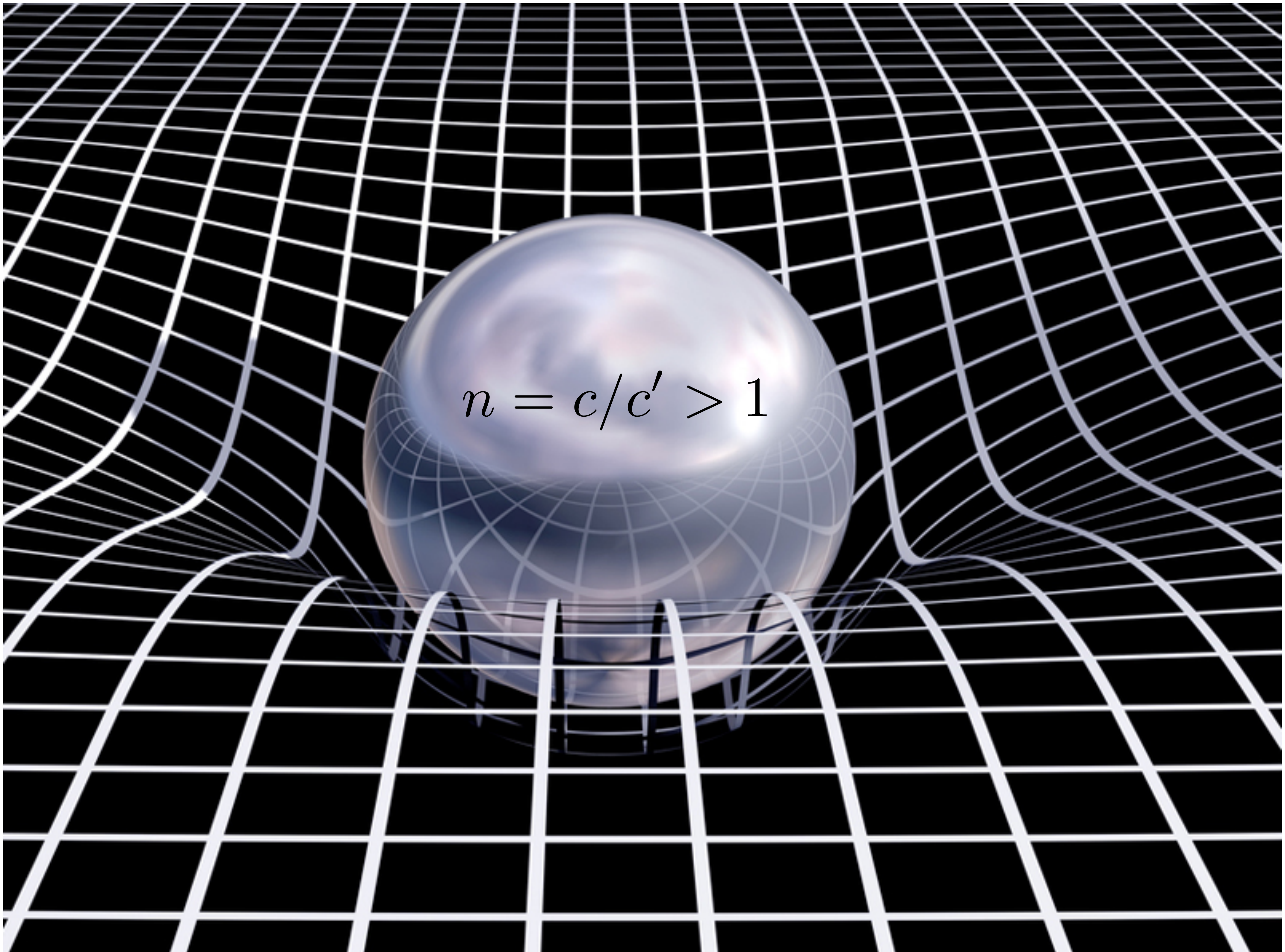


DEFLECTION OF LIGHT IN GENERAL RELATIVITY

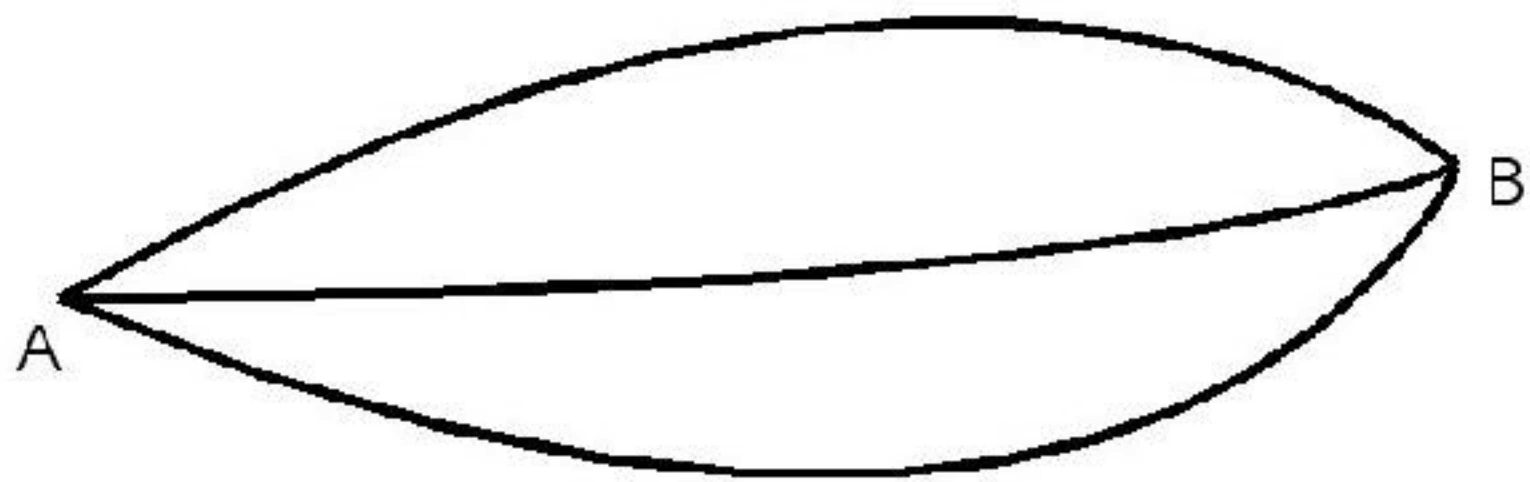
- We will now repeat the calculation of the deflection angle in the context of a locally curved space-time
- Assumptions:
  - the deflection occurs in small region of the universe and over time-scales where the expansion of the universe is not relevant
  - the weak-field limit can be safely applied: $|\Phi|/c^2 \ll 1$
  - perturbed region can be described in terms of an effective diffraction index
  - Fermat principle

DEFLECTION OF LIGHT IN GENERAL RELATIVITY



DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Travel time $= \int \frac{n}{c} \mathrm{d}l$



Fermat principle: $$\delta \int_A^B n dl = 0$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

How to define the effective diffraction index?

absence of lens = unperturbed space-time described by the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (\mathrm{d}x^0)^2 - (\mathrm{d}\vec{x})^2 = c^2\mathrm{d}t^2 - (\mathrm{d}\vec{x})^2$$

effective diffraction index >1 = perturbed space-time, described by the perturbed metric

$$\eta_{\mu\nu} \to g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1-\frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1-\frac{2\Phi}{c^2}) & 0 \\ 0 & 0 & 0 & -(1-\frac{2\Phi}{c^2}) \end{pmatrix}$$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2\Phi}{c^2}\right)(\mathrm{d}\vec{x})^2$$

SCHWARZSCHILD METRIC (STATIC AND SPHERICALLY SYMMETRIC)

$$ds^2 = \left(1 - \frac{2GM}{Rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{Rc^2}\right)^{-1} dR^2 - R^2(\sin^2\theta d\phi^2 + d\theta^2)$$

$$R = \sqrt{1 + \frac{2GM}{rc^2}}r$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\phi$$

$$dl^2 = [dr^2 + r^2(\sin^2\theta d\phi^2 + d\theta^2)]$$

$$\mathrm{d}s^2 = \left(\frac{1 - GM/2rc^2}{1 + GM/2rc^2}\right)^2 c^2\mathrm{d}t^2 - \left(1 + \frac{GM}{2rc^2}\right)^4 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

SCHWARZSCHILD METRIC IN THE WEAK FIELD LIMIT

$$\Phi/c^2 = -GM/rc^2 \ll 1$$

$$\begin{aligned}
\left(\frac{1 - GM/2rc^2}{1 + GM/2rc^2}\right)^2 &\approx \left(1 - \frac{GM}{2rc^2}\right)^4 \\
&\approx \left(1 - \frac{2GM}{rc^2}\right) \\
&= \left(1 + \frac{2\Phi}{c^2}\right)
\end{aligned}
\qquad
\begin{aligned}
\left(1 + \frac{GM}{2rc^2}\right)^4 &\approx \left(1 + 2\frac{GM}{rc^2}\right) \\
&= \left(1 - \frac{2\Phi}{c^2}\right).
\end{aligned}$$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

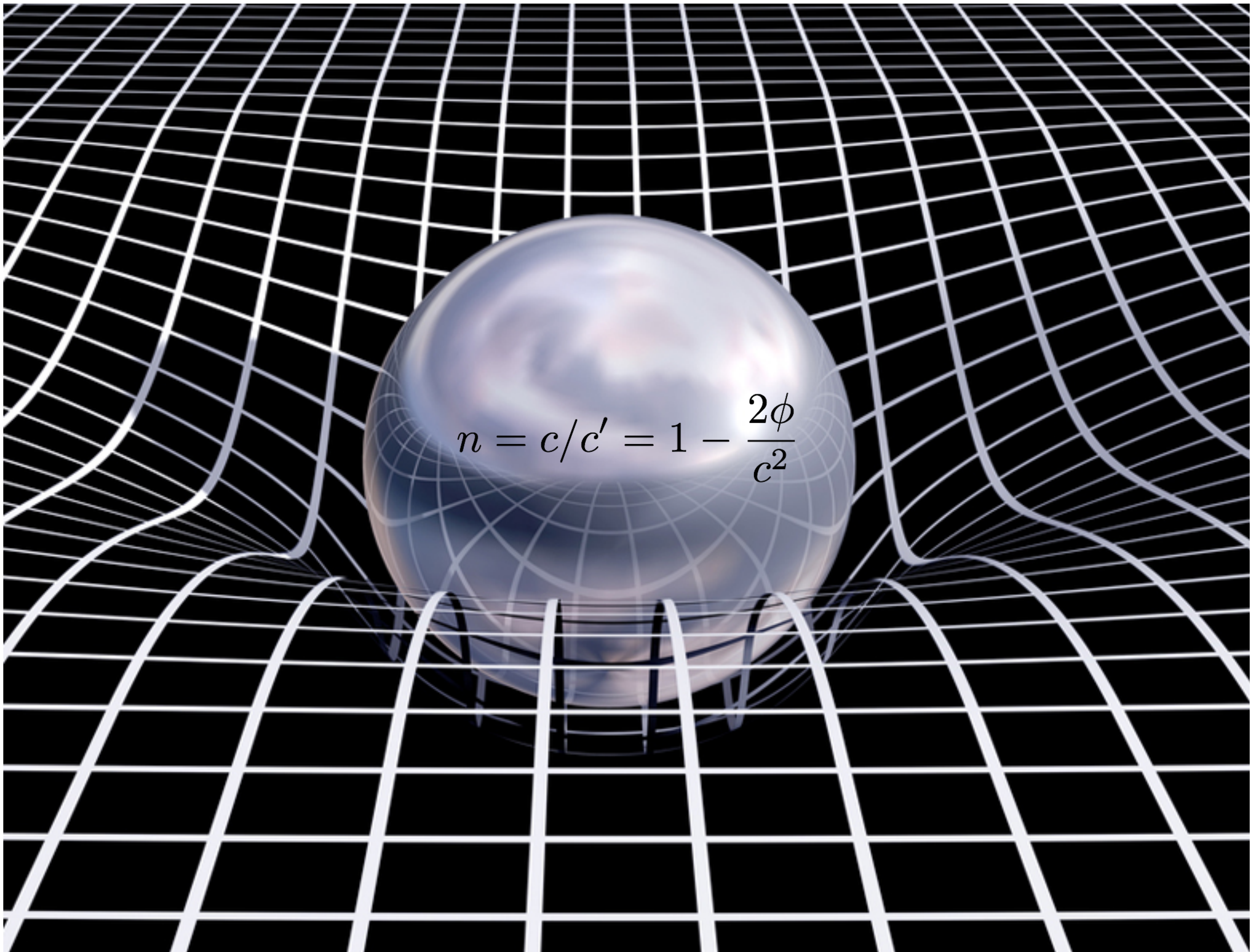
How to define the effective diffraction index?

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 = \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \approx c\left(1 + \frac{2\Phi}{c^2}\right)$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY



DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle

$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

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$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right|\mathrm{d}\lambda$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle

$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle

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generalized coordinate

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

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$$\boxed{n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \equiv L(\dot{\vec{x}}, \vec{x}, \lambda)}$$

Langrangian!

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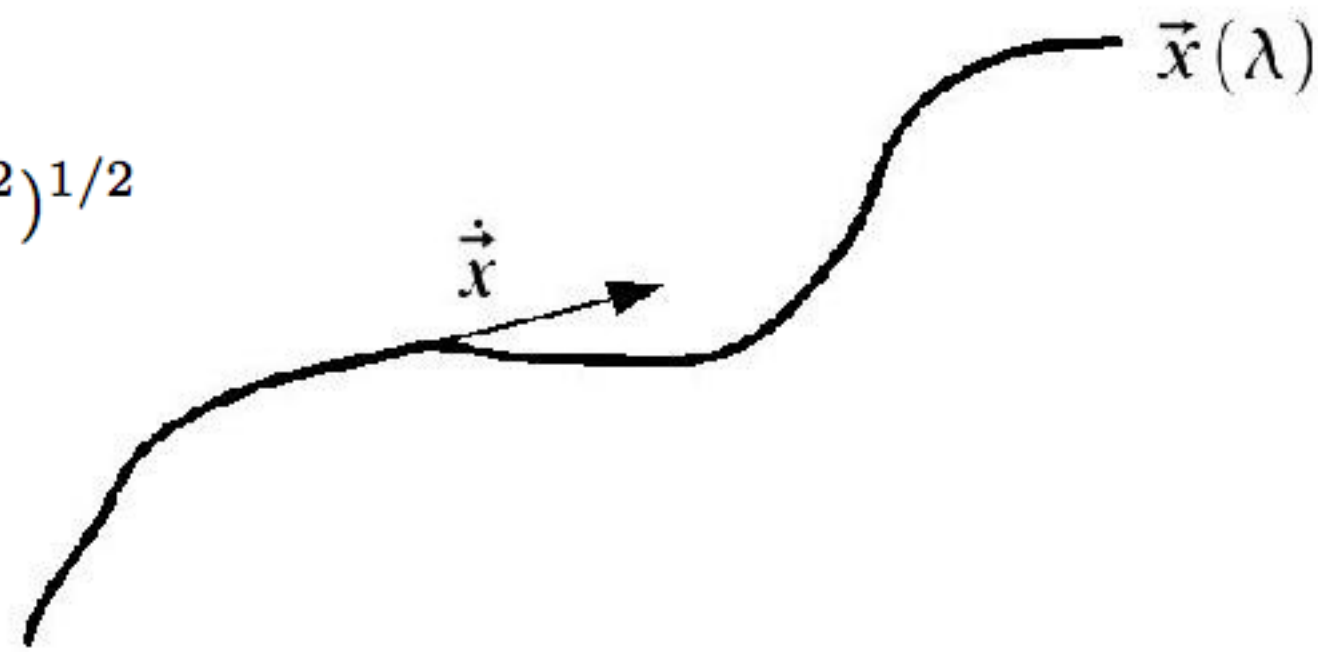

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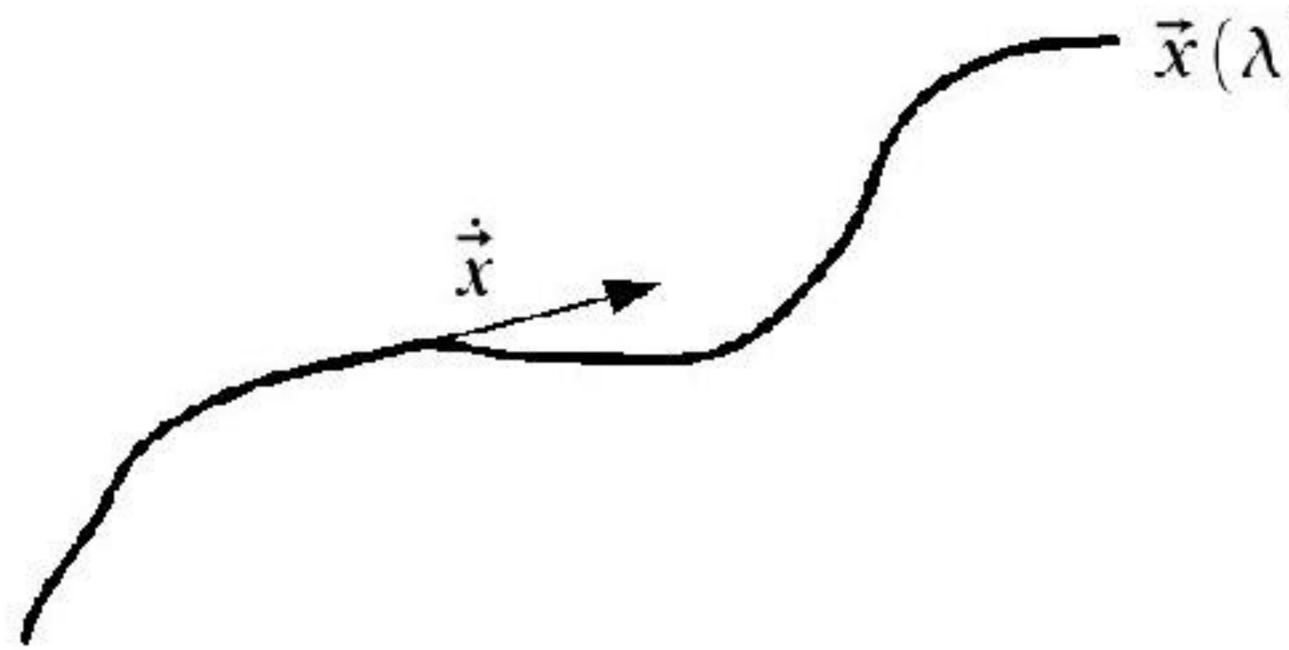

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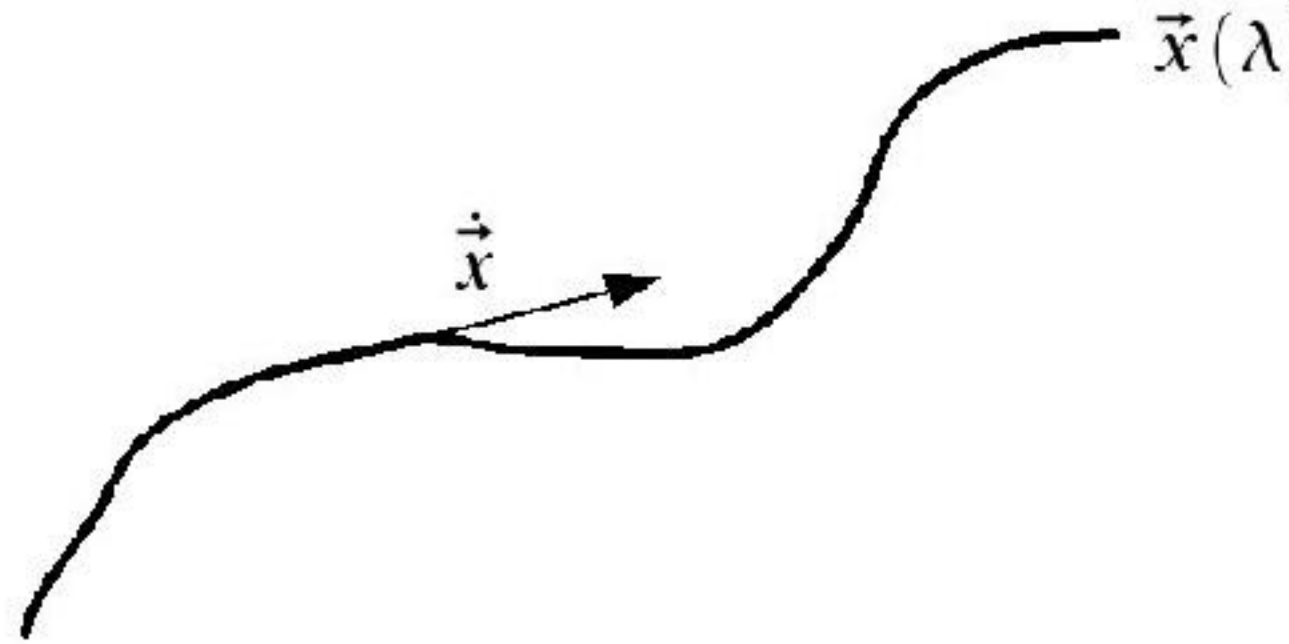

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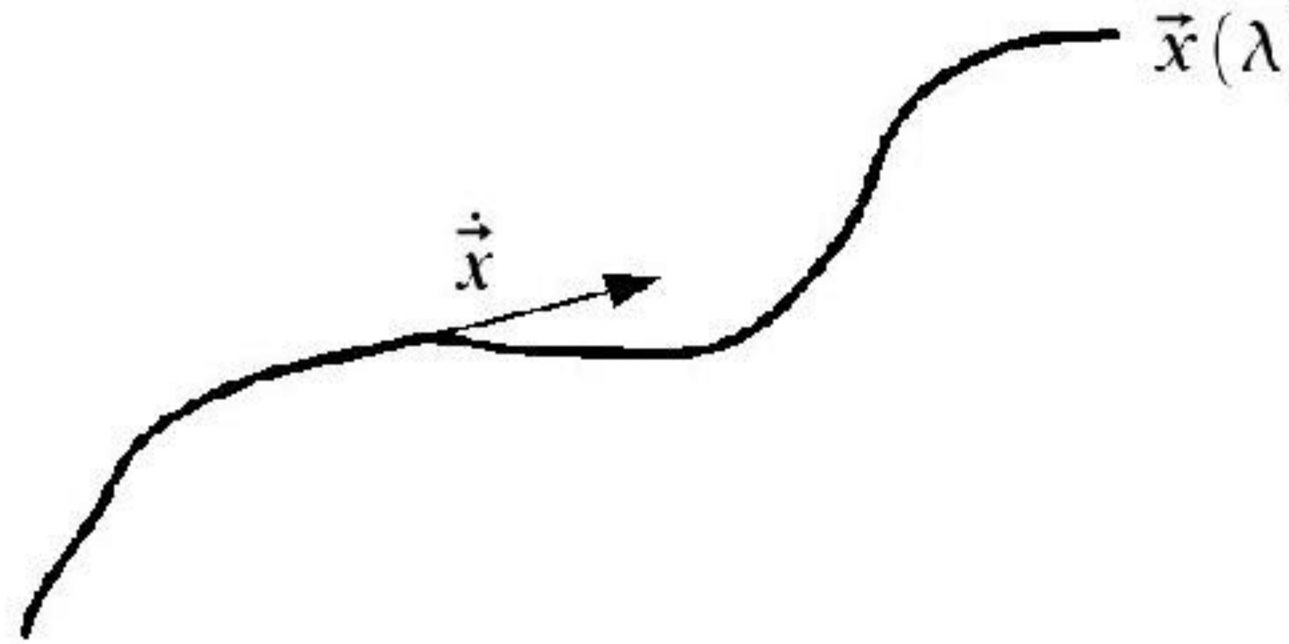

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$$\Rightarrow n\dot{\vec{e}} = \vec{\nabla} n - \vec{e}(\vec{\nabla} n \cdot \vec{e})$$

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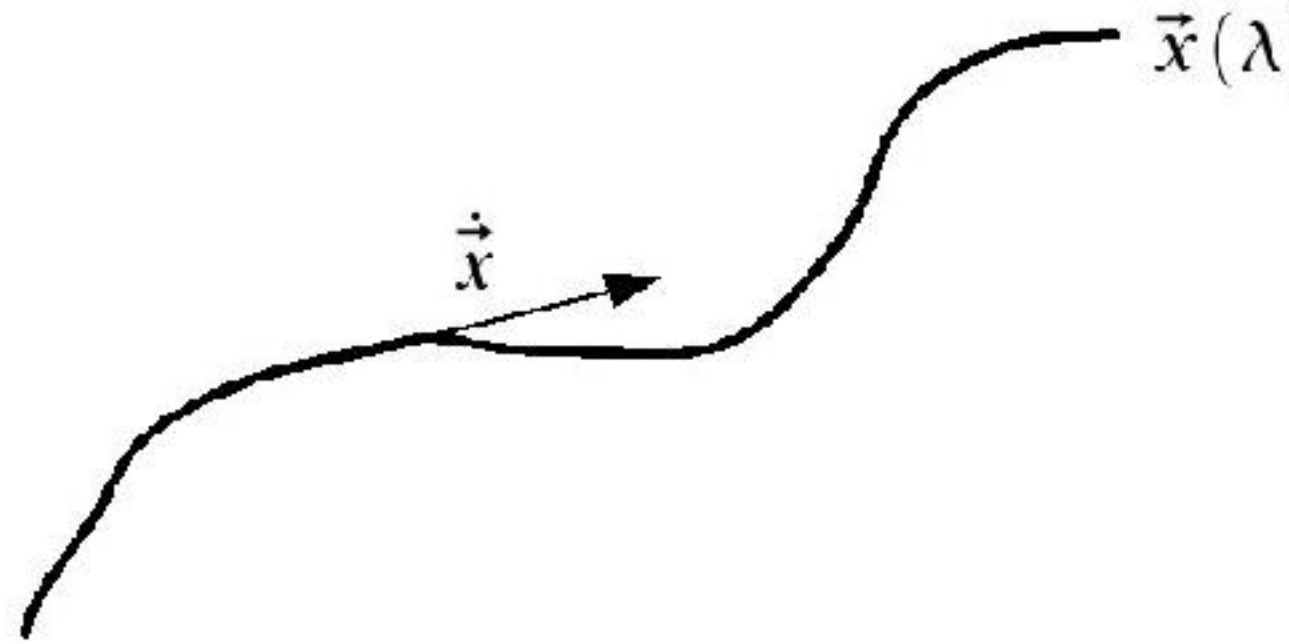

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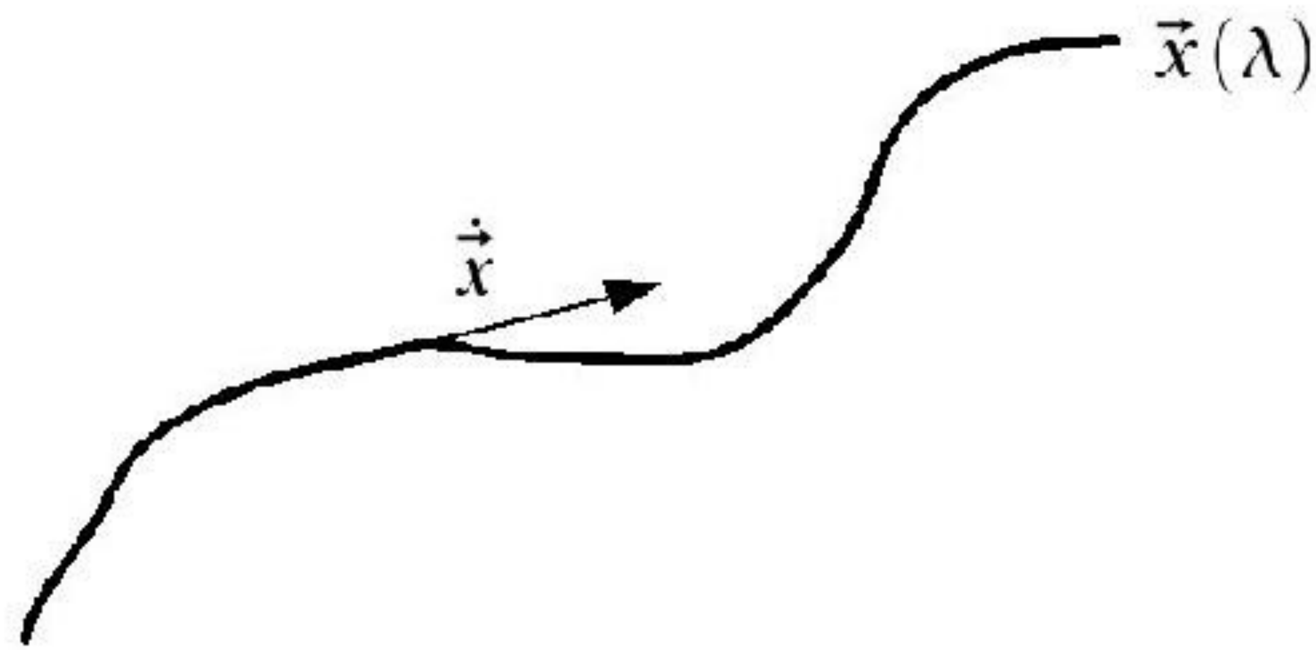

$$n = 1 - \frac{\phi}{c^2} \qquad \frac{\phi}{c^2} \ll 1$$

$$\ln n \approx -2\frac{\phi}{c^2}$$

$$\dot{\vec{e}} \approx -\frac{2}{c^2}\vec{\nabla}_\perp \Phi$$

$$\boxed{\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \mathrm{d}\lambda}$$

Deflection angle

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

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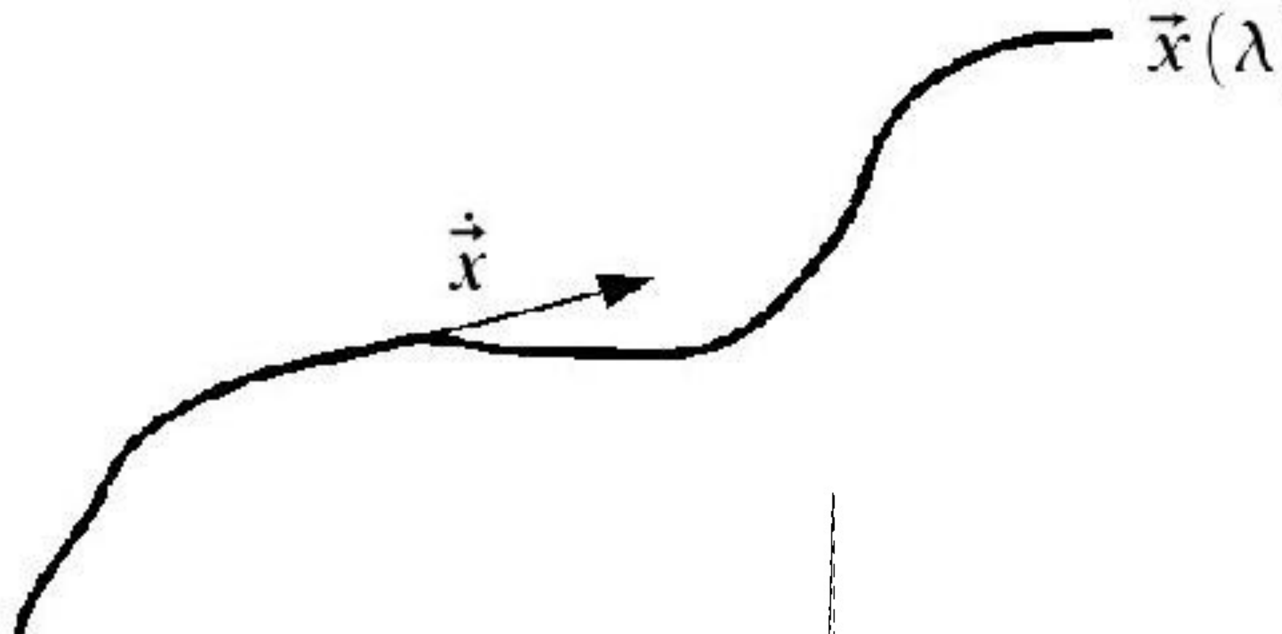
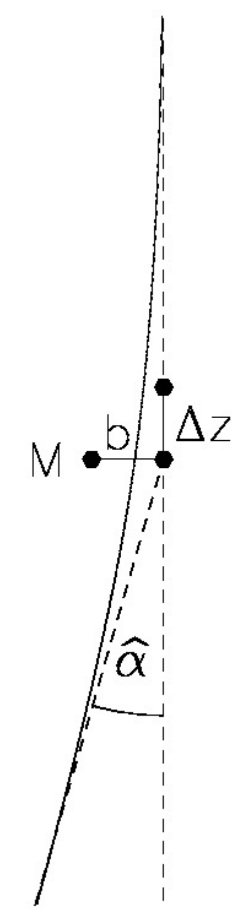


As it is written, this equation is not useful, as we would have to integrate over the actual light path.

Let's assume that the deflection is small. We can integrate the potential along the unperturbed path (Born approximation):



$$\boxed{\hat{\vec{\alpha}}(b) = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi \mathrm{d}z}$$

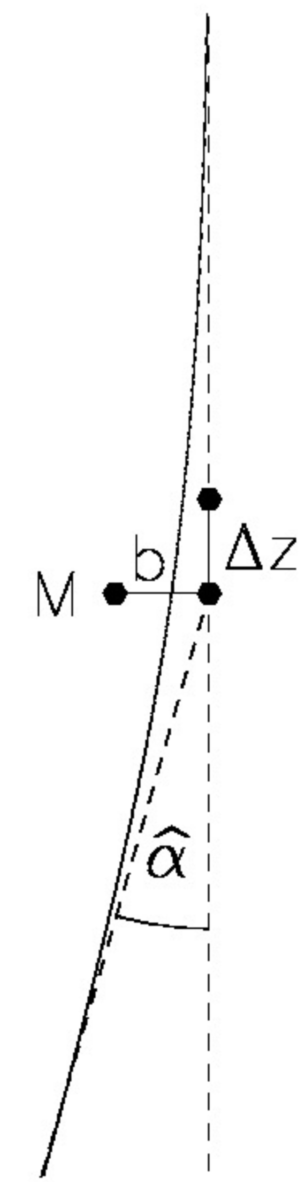
A PARTICULAR CASE: THE POINT MASS

$$\phi = -\frac{GM}{r}$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{b^2 + z^2}$$

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \vec{\hat{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{dz}{(b^2 + z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2 (b^2 + z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \end{aligned}$$



A LIGHT RAY GRAZING THE SURFACE OF THE SUN

General relativity:

$$\hat{\alpha} = \frac{4GM_\odot}{c^2 R_\odot} = 1.75''$$

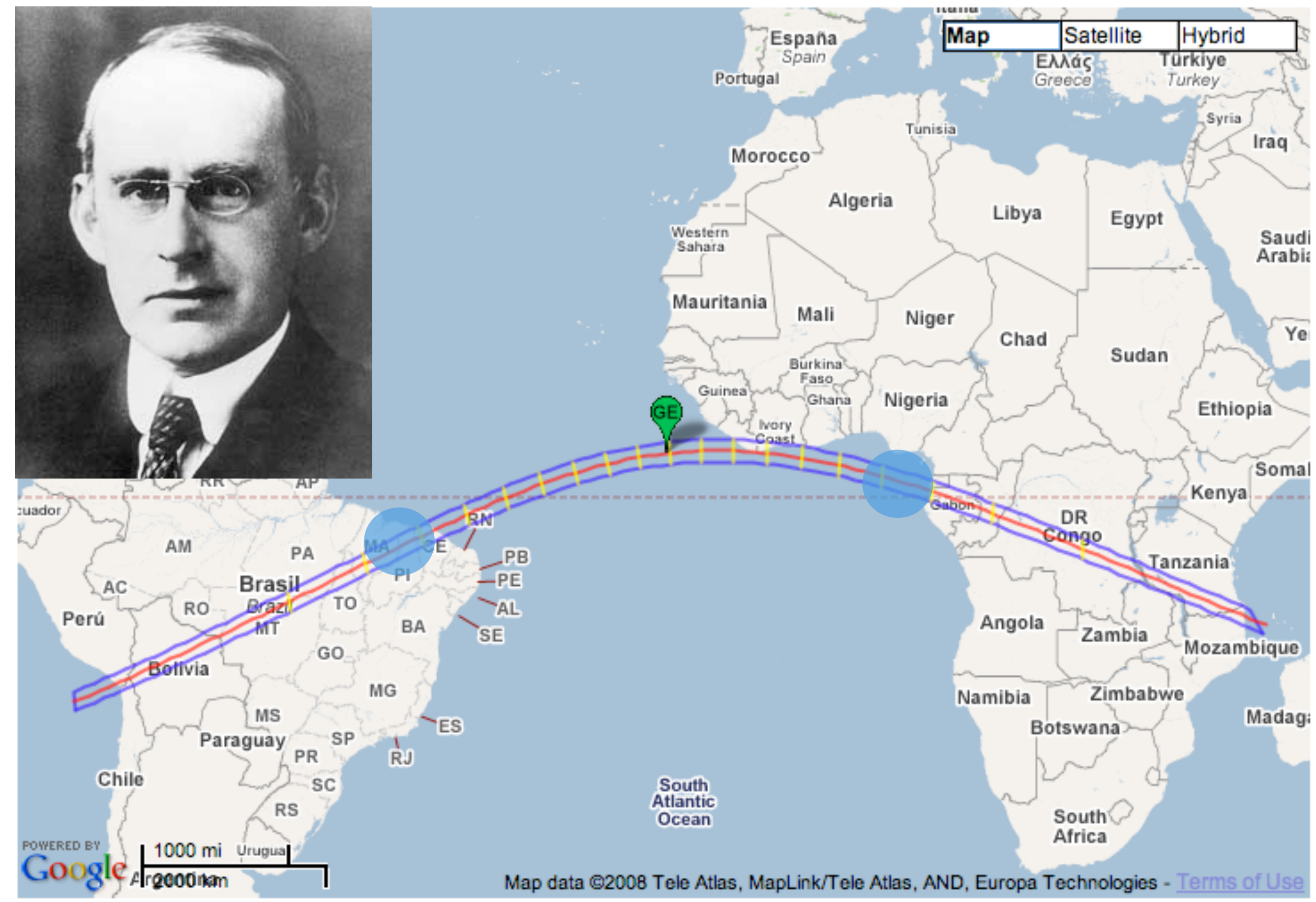
Newtonian gravity and corpuscolar light:

$$\hat{\alpha} = \frac{2GM_\odot}{c^2 R_\odot} = 0.875''$$

*The reason for the factor of 2 difference is that **both the space and time coordinates are bent** in the vicinity of massive objects — it is four-dimensional space–time which is bent by the Sun.*

EDDINGTON EXPEDITIONS

- In 1919 Eddington organized two expeditions to observe a total solar eclipse (Principe Island and Sobral)
- The goal was to measure the lensing effect of the sun on background stars
- Very conveniently, the sun was well aligned with the Iades open cluster
- During the eclipse the expedition from Principe registered a shift in the apparent position of stars with respect to their night-time positions, which resulted to be consistent with the GR predictions
- The Sobral expedition measured a smaller deflection but this was interpreted as the result of a technical problem.



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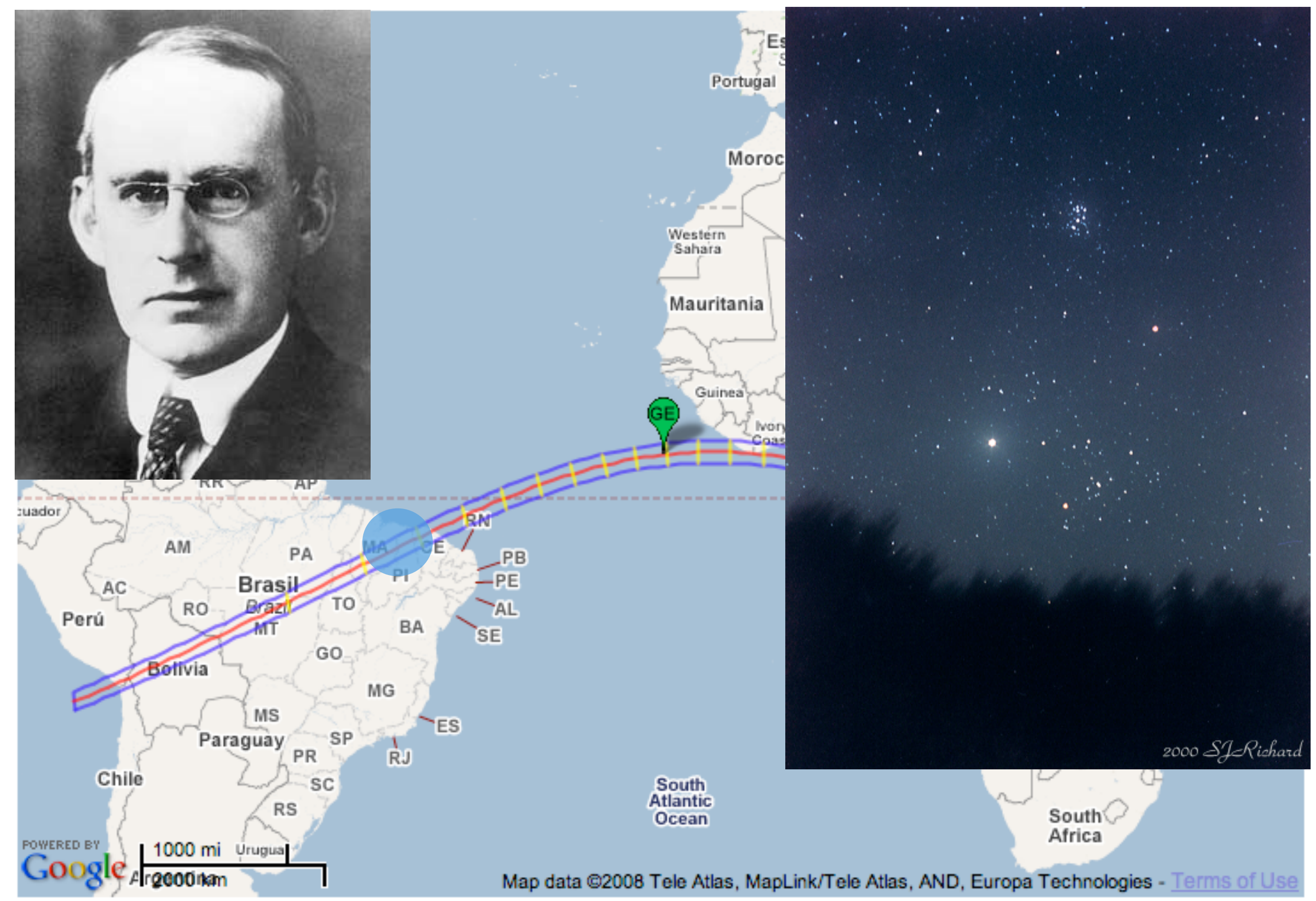

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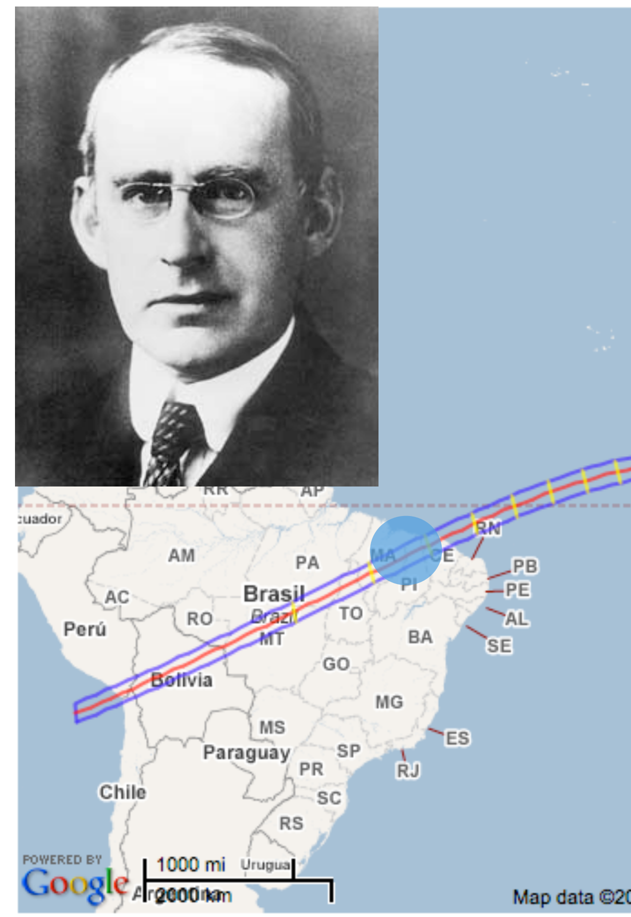

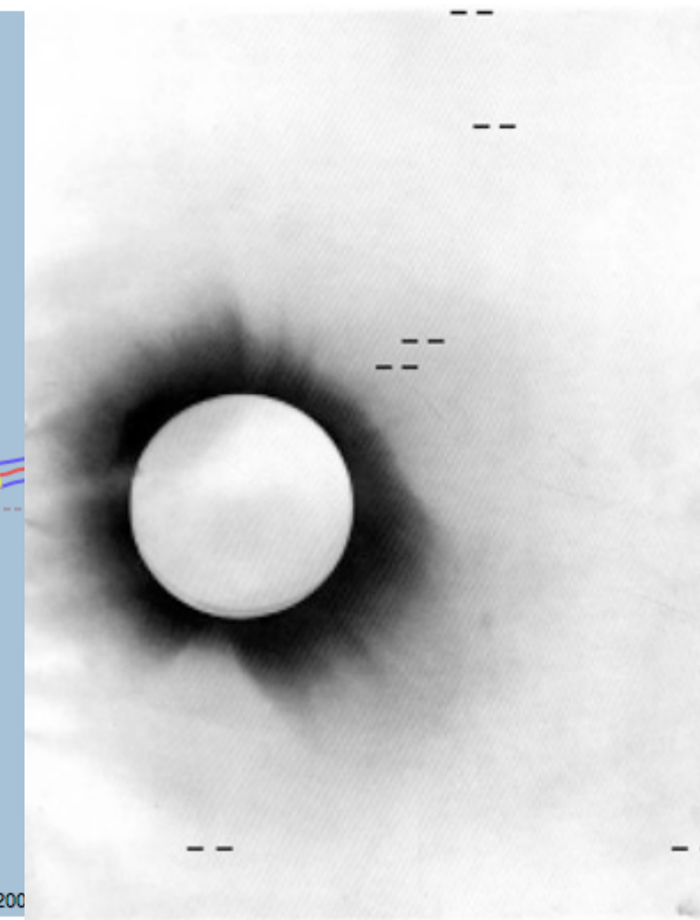

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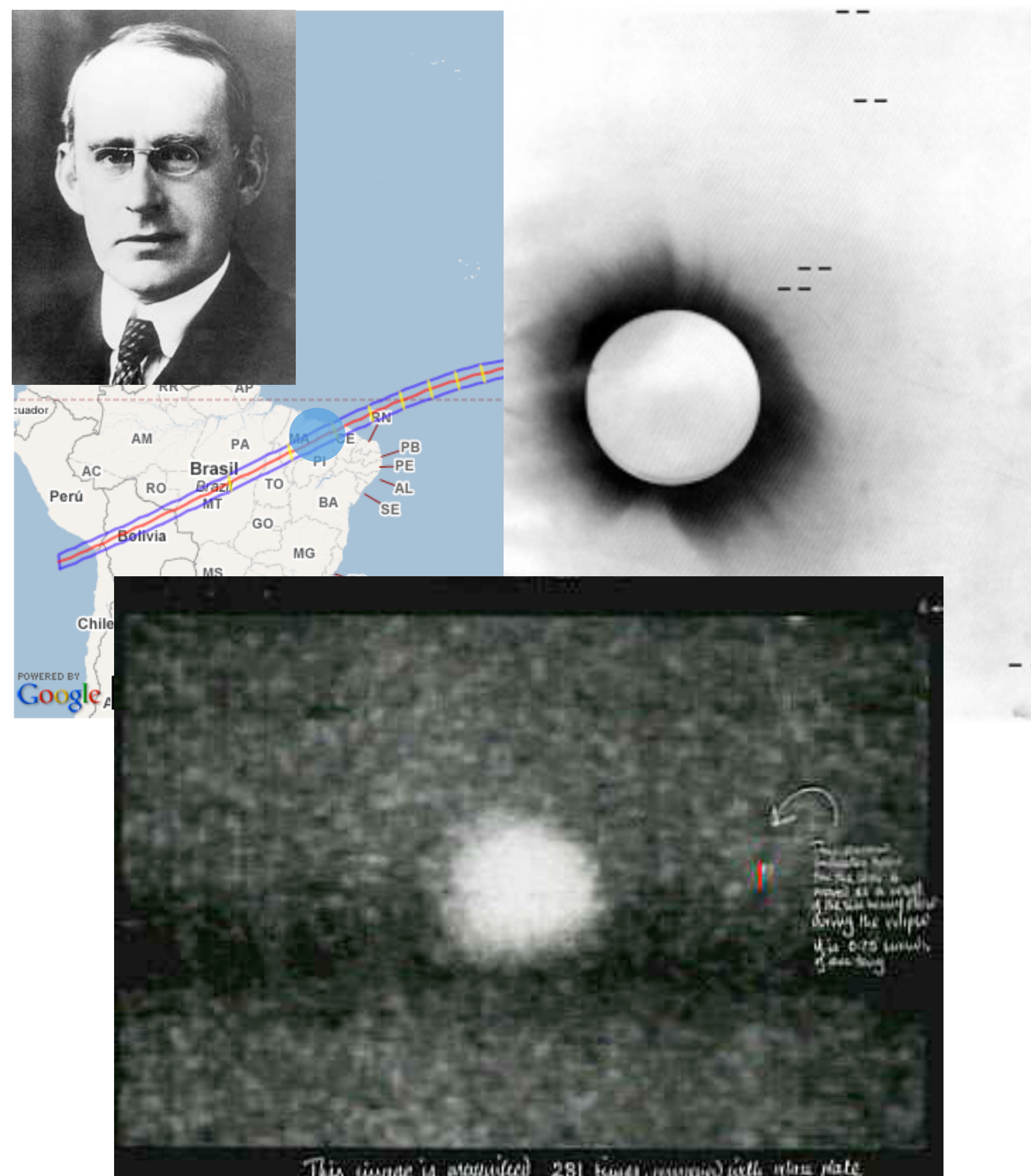

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