i.e. a source at $y = 1$ has two images at

$$x_\pm = \frac{1 \pm \sqrt{5}}{2} \ , \tag{3.14}$$

and their magnifications are

$$\mu_\pm = \left[1 - \left(\frac{2}{1 \pm \sqrt{5}}\right)^4\right]^{-1} \ . \tag{3.15}$$

For general source positions,

$$\begin{aligned} \mu_\pm &= \left[1 - \left(\frac{1}{x_\pm}\right)^4\right]^{-1} \\ &= \frac{x_\pm^4}{x_\pm^4 - 1} = \frac{1}{2} \pm \frac{y^2 + 2}{2y\sqrt{y^2+4}} \ . \end{aligned} \tag{3.16}$$

Note that $\lim_{y\to\infty} \mu_- = 0$ and that $\lim_{y\to\infty} \mu_+ = 1$: even if the lens equation has always two solutions, for large angular separations between the source and the lens, one image desappears because it is demagnified. The other is completely undistiguishable from the source because it has the same flux and the same position.

The total magnification of a point source by a point mass is thus

$$\mu = |\mu_+| + |\mu_-| = \frac{y^2 + 2}{y\sqrt{y^2+4}} \ , \tag{3.17}$$

and the magnification ratio of the two images is

$$|\frac{\mu_-}{\mu_+}| = \left(\frac{y - \sqrt{y^2+4}}{y + \sqrt{y^2+4}}\right)^2 = \left(\frac{x_-}{x_+}\right) \ . \tag{3.18}$$

If $\beta = \theta_E$, $y = 1$ and the total magnification is $\mu = 1.17 + 0.17 = 1.34$. In terms of magnitudes, this correspond to $\Delta m = -2.5 \log \mu \sim 0.3$. The image forming at $x_+$ contributes for $\sim 87\%$ of the total magnification.

Lensing by point masses on point sources will be discussed in detail in a following chapter. However, we can already answer to the question: how can lensing by a point mass be detected? Unless the lens is more massive than $10^6\,M_\odot$ (for a source at cosmological distance), the angular separation between multiple images is too small to be resolved. However, the magnification effect will be detectable in many cases if the source is moving relative to the lens (for example, a star in the large magellanic cloud is in relative motion with respect to a star in the halo of our galaxy). Thus, since the magnification changes as a function of the angular separation between source and lens, the lensing effect will induce a time variability in the light curve of the source.

## 3.2 Axially symmetric lenses

The main advantage of using axially symmetric lenses is that their surface density is independent on the position angle with respect to lens center. If we choose the optical axis such that it intercepts the lens plane in the lens center, this implies that $\Sigma(\vec{\xi}) = \Sigma(|\vec{\xi}|)$. The lensing equations therefore reduce to a one-dimensional form, since all the light rays from a (point) source lie on the same plane passing through the center of the lens, the source and the observer.

The deflection angle for an axially symmetric lens was found to be

$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi} . \tag{3.19}$$

If want to use adimensional quantities:

$$\begin{aligned}
\alpha(x) &= \frac{D_\mathsf{L} D_\mathsf{LS}}{\xi_0 D_\mathsf{S}} \hat{\alpha}(\xi_0 x) \\
&= \frac{D_\mathsf{L} D_\mathsf{LS}}{\xi_0 D_\mathsf{S}} \frac{4GM(\xi_0 x)}{c^2\xi} \frac{\pi\xi_0}{\pi\xi_0} \\
&= \frac{M(\xi_0 x)}{\pi\xi_0^2\Sigma_\mathrm{cr}} \frac{1}{x} \equiv \frac{m(x)}{x} ,
\end{aligned} \tag{3.20}$$

where we have introduced the *dimensionless mass* $m(x)$. Note that

$$\alpha(x) = \frac{2}{x}\int_0^x x'\kappa(x')\mathrm{d}x' \Rightarrow m(x) = 2\int_0^x x'\kappa(x')\mathrm{d}x' . \tag{3.21}$$

The lens equation (2.9) then becomes

$$y = x - \frac{m(x)}{x} . \tag{3.22}$$

Now, we derive formulas for several lensing quantities. To do that, we need to write the deflection angle as a vector. For an axially symmetric lens, the deflection angle points towards the lens center. Then,

$$\vec{\alpha}(\vec{x}) = \frac{m(\vec{x})}{x^2}\vec{x} , \tag{3.23}$$

where $\vec{x} = (x_1, x_2)$.
By differentiating we obtain:

$$\frac{\partial\alpha_1}{\partial x_1} = \frac{\mathrm{d}m}{\mathrm{d}x}\frac{x_1^2}{x^3} + m\frac{x_2^2 - x_1^2}{x^4} , \tag{3.24}$$

$$\frac{\partial\alpha_2}{\partial x_2} = \frac{\mathrm{d}m}{\mathrm{d}x}\frac{x_2^2}{x^3} + m\frac{x_1^2 - x_2^2}{x^4} , \tag{3.25}$$

$$\frac{\partial\alpha_1}{\partial x_2} = \frac{\partial\alpha_2}{\partial x_1} = \frac{\mathrm{d}m}{\mathrm{d}x}\frac{x_1 x_2}{x^3} - 2m\frac{x_1 x_2}{x^4} , \tag{3.26}$$

which immediately give the elements of the Jacobian matrix:

$$\begin{aligned}
A &= I - \frac{m(x)}{x^4}\begin{pmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} \\
&\quad - \frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3}\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} .
\end{aligned} \tag{3.27}$$

This permits us to obtain the following expressions for the convergence and the shear components:

$$\kappa(x) = \frac{1}{2x}\frac{\mathrm{d}m(x)}{\mathrm{d}x} , \tag{3.28}$$

$$\gamma_1(x) = \frac{1}{2}(x_2^2 - x_1^2)\left(\frac{2m(x)}{x^4} - \frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3}\right) , \tag{3.29}$$

$$\gamma_2(x) = x_1x_2\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3} - \frac{2m(x)}{x^4}\right) . \tag{3.30}$$

From these relations,

$$\gamma(x) = \frac{m(x)}{x^2} - \kappa(x) \ . \tag{3.31}$$

Since $m(x) = 2\int_0^x x'\kappa(x')\mathrm{d}x'$, we see that

$$\frac{m(x)}{x^2} = 2\pi\frac{\int_0^x x'\kappa(x')\mathrm{d}x'}{\pi x^2} = \overline{\kappa}(x) \ . \tag{3.32}$$

where $\overline{\kappa}(x) = m(x)/x^2$ is the *mean surface mass density* within $x$. Eq. 3.31 then reduces to

$$\gamma(x) = \overline{\kappa}(x) - \kappa(x) \tag{3.33}$$

The Jacobian determinant of the lens mapping is

$$\begin{aligned} \det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left(1 - \frac{m(x)}{x^2}\right)\left[1 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{m(x)}{x}\right)\right] \\ &= \left(1 - \frac{m(x)}{x^2}\right)\left(1 + \frac{m(x)}{x^2} - 2\kappa(x)\right) \\ &= \left(1 - \frac{\alpha(x)}{x}\right)\left(1 - \frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right) \ . \end{aligned} \tag{3.34}$$

**Critical lines and caustics:** Since the critical lines arise where $\det A = 0$, Eq. (3.34) implies that axially symmetric lenses with monotonically increasing $m(x)$ have at most two critical lines, where $m(x)/x^2 = 1$ and $\mathrm{d}(m(x)/x)/\mathrm{d}x = \mathrm{d}y/\mathrm{d}x = 1$. Both these conditions define circles on the lens plane (see Fig. 3.1). The critical line along which $m(x)/x^2 = 1$ is the tangential one: any vector which is tangential to this line is an eigenvector with zero eigenvalue of the Jacobian matrix. On the other hand, given that any vector perpendicular to the critical line where $\mathrm{d}(m(x)/x)/\mathrm{d}x = 1$ is also an eigenvector with zero eigenvalue, this line is the radial critical line. This can be seen as follows. Consider a point $(x, 0)$ on a critical line. Although this point has been chosen to lay on the $x_1$-axis, this discussion can be generalized to any other critical point, since the reference frame can be arbitrarily chosen. The Jacobian at $(x, 0)$ is readily derived from Eq. 3.27:

$$A(x,0) = I - \frac{m(x)}{x^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{3.35}$$

Let consider a vector whose components are $(0, a)$ at $(x, 0)$. This vector is clearly tangential to the critical line at $(x, 0)$. Through the lens mapping, it is mapped onto

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A(x,0)\begin{pmatrix} 0 \\ a \end{pmatrix} \tag{3.36}$$

Clearly the last term in Eq. 3.35 returns the null vector when applied to $(0, a)$. If $(x, 0)$ lays on the tangential critical line, then $(1 - m(x)/x^2) = 0$ and

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(1 - \frac{m}{x^2}\right)\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ . \tag{3.37}$$

Thus $(0, a)$ is an eigenvector of $A$ with 0 eigenvalue.

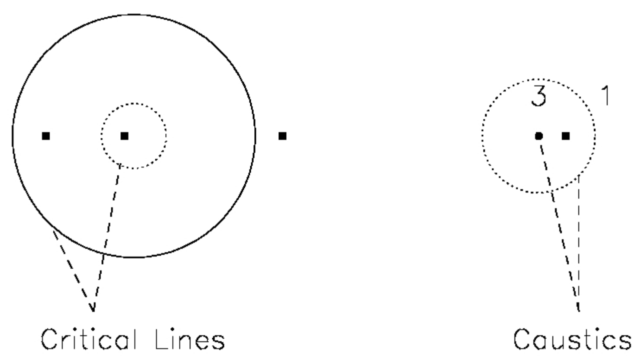


**Figure 3.1:** Imaging of a point source by a non-singular, circularly-symmetric lens. Left: image positions and critical lines; right: source position and corresponding caustics. From Narayan & Bartelmann (1995).

Consider now a vector $(b, 0)$, normal to the critical line at $(x, 0)$. Mapping it to the source plane we obtain:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A(x,0)\begin{pmatrix} b \\ 0 \end{pmatrix} = \left(1 + \frac{m(x)}{x^2} - \frac{1}{x}\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)\begin{pmatrix} b \\ 0 \end{pmatrix} . \tag{3.38}$$

If $(x, 0)$ lays on the radial critical line, then $[1 + m(x)/x^2 - m'(x)/x] = 0$, thus $(b, 0)$ is an eigenvector of $A$ is 0 eigenvalue.

From the lens equation it can be easily seen that all the points along the tangential critical line are mapped on the point $y = 0$ on the source plane. Indeed:

$$y = x\left(1 - \frac{m}{x^2}\right) = 0 . \tag{3.39}$$

if $x$ indicates a tangential critical point. Therefore, axially symmetric models have point tangential caustics. On the other hand, the points along the radial critical line are mapped onto a circular caustic on the source plane.

**Image distortions near the critical lines:** Let us consider now how the images are distorted near the critical lines. Consider a point $(x_c, 0)$ very close to the tangential critical line. At this point,

$$\frac{m(x)}{x^2} = 1 - \delta , \tag{3.40}$$

where $\delta \ll 1$.

Using Eq. 3.35, we see that near the tangential critical line the Jacobian is approximated by

$$A(x_c, 0) \simeq \begin{pmatrix} 2 - m'/x_c & 0 \\ 0 & \delta \end{pmatrix} . \tag{3.41}$$

In the first element of the matrix we have neglected $\delta$, being it small. Consider an ellipse around $\vec{x}_c = (x_c, 0)$,

$$\vec{c}(\phi) = \vec{x}_c + \begin{pmatrix} \rho_1 \cos\phi \\ \rho_2 \sin\phi \end{pmatrix} . \tag{3.42}$$

Through the lens mapping, the source of this ellipse is

$$\vec{d}(\phi) = \vec{y}_c + \begin{pmatrix} \rho_1 (2 - m'/x_c) \cos\phi \\ \rho_2 \delta \sin\phi \end{pmatrix} . \tag{3.43}$$

Suppose that $\vec{d}(\phi)$ is a circle, i.e. $\rho_1(2 - m'/x_c) = \rho_2 \delta$. Then,

$$\frac{\rho_2}{\rho_1} = \frac{2 - m'/x_c}{\delta} \gg 1 . \tag{3.44}$$

Thus, the ellipse is strongly elongated along the $x_2$ direction.

On the contrary, suppose that $(x_c, 0)$ is very close to the radial critical line. In this case,

$$\frac{m'}{x_c} - \frac{m}{x_c^2} = 1 - \delta , \tag{3.45}$$

with $\delta \ll 1$. The Jacobian matrix at $(x_c, 0)$ is then

$$A(x_c, 0) \simeq \begin{pmatrix} \delta & 0 \\ 0 & 1 - m/x_c^2 \end{pmatrix} . \tag{3.46}$$

The source corresponding to the ellipse in Eq. 3.42 is

$$\vec{d}(\phi) = \vec{y}_c + \begin{pmatrix} \rho_1 \delta \cos\phi \\ \rho_2 (1 - m/x_c^2) \sin\phi \end{pmatrix} . \tag{3.47}$$

Thus, if $\vec{d}(\phi)$ is a circle,

$$\frac{\rho_1}{\rho_2} = \frac{1 - m/x_c^2}{\delta} \gg 1 , \tag{3.48}$$

and the ellipse is now strongly elongated along the $x_1$ direction.

Summarizing, any image near the tangential critical curve is strongly distorted tangentially to the critical curve, while any image near the radial critical curve is strongly distorted normally to the critical curve. The discussion done here is still neglecting high order lensing effects. In the real world, images near the critical lines are not ellipses. Rather they are bent to form complex shapes, like arcs, etc. Such distortions are more evident if the sources are extended. Fig. 3.2 shows the images of two extended sources lensed by the same model as in Fig. 3.1. One source is located close to the point-like caustic in the center of the lens. It is imaged onto the two long, tangentially oriented arcs close to the outer critical curve and the very faint image at the lens center. The other source is located on the outer caustic and forms a radially elongated image which is composed of two merging images, and a third tangentially oriented image outside the outer critical line.

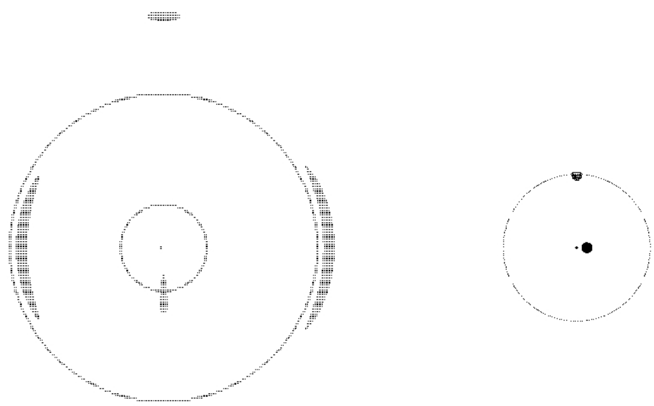

**Figure 3.2:** Imaging of an extended source by a non-singular circularly-symmetric lens. A source close to the point caustic at the lens center produces two tangentially oriented arc-like images close to the outer critical curve, and a faint image at the lens center. A source on the outer caustic produces a radially elongated image on the inner critical curve, and a tangentially oriented image outside the outer critical curve. From Narayan & Bartelmann (1995).

**Tangential and radial magnification of the images:** As was pointed out in the previous chapter, the eigenvalues of the Jacobian matrix give the inverse magnification of the image along the tangential and radial directions. Fig. (3.3) illustrates an infinitesimal source of diameter $\delta$ at position $y$ and its image, which is an ellipse, whose minor and major axes are $\rho_1$ and $\rho_2$ respectively, at position $x$. With respect to the origin of the reference frame on the source plane, the circular source subtends an angle $\phi = \delta/y$. Due to the axial symmetry of the lens, $\phi = \rho_2/x$. Using the lens equation, we thus obtain

$$\frac{\delta}{\rho_2} = 1 - \frac{m(x)}{x^2} \ . \tag{3.49}$$

The lens mapping gives $\delta = \rho_1 (\mathrm{d}y/\mathrm{d}x)$, from which

$$\frac{\delta}{\rho_1} = 1 + \frac{m(x)}{x^2} - 2\kappa(x) \tag{3.50}$$

This means that the image is stretched in the tangential direction by a factor $[1 - m(x)/x^2]^{-1}$ and in the radial direction by $[1 + m(x)/x^2 - 2\kappa(x)]^{-1}$.

**Generalities about the images:** As discussed previously, if the lens is strong, multiple images can be formed of the same source. The number of these images depends on the position of the source with respect to the caustics. Sources which lie within the radial