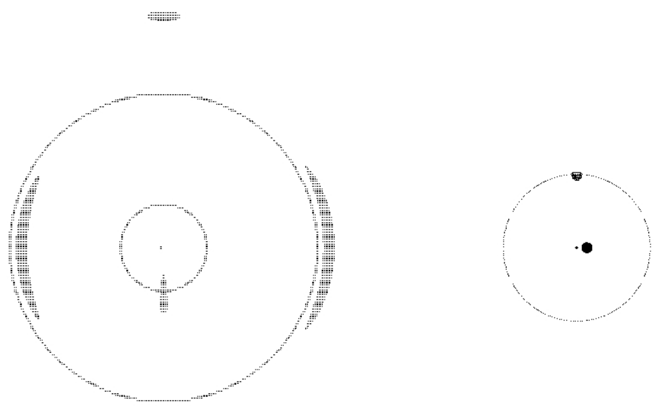

Figure 3.2: Imaging of an extended source by a non-singular circularly-symmetric lens. A source close to the point caustic at the lens center produces two tangentially oriented arc-like images close to the outer critical curve, and a faint image at the lens center. A source on the outer caustic produces a radially elongated image on the inner critical curve, and a tangentially oriented image outside the outer critical curve. From Narayan & Bartelmann (1995).

Tangential and radial magnification of the images: As was pointed out in the previous chapter, the eigenvalues of the Jacobian matrix give the inverse magnification of the image along the tangential and radial directions. Fig. (3.3) illustrates an infinitesimal source of diameter δ at position y and its image, which is an ellipse, whose minor and major axes are ρ_1 and ρ_2 respectively, at position x. With respect to the origin of the reference frame on the source plane, the circular source subtends an angle $\phi = \delta/y$. Due to the axial symmetry of the lens, $\phi = \rho_2/x$. Using the lens equation, we thus obtain

$$\frac{\delta}{\rho_2} = 1 - \frac{m(x)}{x^2} \ . \tag{3.49}$$

The lens mapping gives $\delta = \rho_1(\mathrm{d}y/\mathrm{d}x)$, from which

$$\frac{\delta}{\rho_1} = 1 + \frac{m(x)}{x^2} - 2\kappa(x) \tag{3.50}$$

This means that the image is stretched in the tangential direction by a factor $[1 - m(x)/x^2]^{-1}$ and in the radial direction by $[1 + m(x)/x^2 - 2\kappa(x)]^{-1}$.

Generalities about the images: As discussed previously, if the lens is strong, multiple images can be formed of the same source. The number of these images depends on the position of the source with respect to the caustics. Sources which lie within the radial

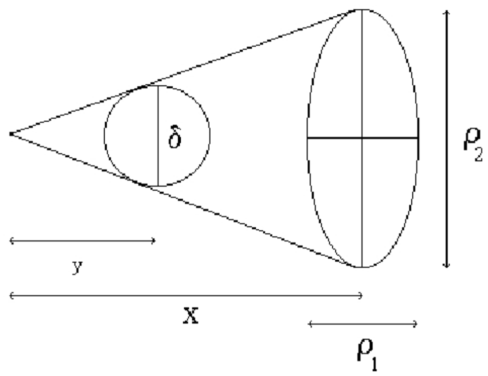

Figure 3.3: Sketch of the mapping of an infinitesimal circular source onto an elliptical image (Figure from Schneider et al., 1992).

caustic produce three images. Sources outside the radial caustic have only one image. This is shown in Fig. 3.1. Since the tangential critical curve does not lead to a caustic curve, but the corresponding caustic degenerates to a single point $\vec{y} = 0$, the tangential critical curves have no influence on the image multiplicity. Thus, pairs of images can only be created or destroyed if the radial critical curve exists.

For non-singular axially symmetric lenses, whose surface density is piecewise continuous and falls off at large radii, such that it is bound, i.e.

$$0 \leq \kappa(x) \leq \kappa_{\max} \tag{3.51}$$

and

$$\lim_{x \to \infty} x\kappa(x) = 0 \; . \tag{3.52}$$

it can be shown that the following properties hold (Schneider et al., 1992):

(1) if the source is at $y > 0$, any image with $x > 0$ is at $x \geq y$. This is easily seen from the lens equation:

$$x = y + \frac{m(x)}{x} \; . \tag{3.53}$$

Being $m(x) \geq 0$ and $x \geq 0$, it follows that $x \geq y$;

(2) for sufficiently large y, there exists a single image. From Eq. 3.52, we see that there must be a constant c and a value a such that for $|x| > a$ $\kappa(x) < c/|x|$. This bounds the mass:

$$\begin{aligned} m(x) &= 2\int_0^x x'\kappa(x')\mathrm{d}x' \\ &= m(a) + 2\int_a^{|x|} x'\kappa(x')\mathrm{d}x' < m(a) + 2c(|x| - a) \; . \end{aligned} \tag{3.54}$$

Thus,

$$\left|\frac{m(x)}{x}\right| < b \,. \tag{3.55}$$

If y is sufficiently large, $y \geq b$, the lens equation tells us that

$$x = y + \frac{m(x)}{x} > 0 \,, \tag{3.56}$$

thus $x \geq y$. Moreover, if $x > a$ then

$$\overline{\kappa} = \frac{m(x)}{x^2} = \frac{m(x)}{x}\frac{1}{x} < \frac{b}{x} \,. \tag{3.57}$$

Thus,

$$\lim_{x\to\infty} y = \lim_{x\to\infty} (1 - \overline{\kappa})x = x \,. \tag{3.58}$$

(3) a lens can produce multiple images if and only if at least at one point $1 - 2\kappa(x) + \overline{\kappa}(x) < 0$: if $1 - 2\kappa(x) + \overline{\kappa}(x) > 0$ throughout, a lens produces no multiple images, since $y(x)$ increases monotonically. If on the other hand, there is a point where $\mathrm{d}y/\mathrm{d}x < 0$, there is at least one local maximum x_1 and one local minimum $x_2 > x_1$ of the curve $y(x)$ since $\mathrm{d}y/\mathrm{d}x \to 1$ for $|x| \to \infty$. For values of y such that $y(x_2) < y < y(x_1)$, there are at least three images;

(4) a necessary condition for multiple images is that $\kappa > 1/2$ at one point in the lens: if $\mathrm{d}y/\mathrm{d}x < 0$ at one point, then $\kappa > (1+\overline{\kappa})/2 \geq 1/2$; a sufficient condition for multiple imaging is that $\kappa > 1$ at one point. Indeed: if κ have a a maximum at one point x_m where $\kappa(x_m) > 1$, then $\overline{\kappa}(x_m) \leq \kappa(x_m)$ and $\mathrm{d}y/\mathrm{d}x < 0$ at x_m. The statement then follows from (3);

(5) if the surface density does not increase with x, $\kappa'(x) \leq 0$, $\kappa(0) > 1$: from (4) we know that it is sufficient that $\kappa > 1$ at one point for having multiple images. On the other hand if $\kappa(0) \leq 1$, then, since $y = x(1 - \overline{\kappa})$, we have for $x \geq 0$: $\mathrm{d}y/\mathrm{d}x = (1 - \overline{\kappa}) - x\overline{\kappa}'$. Since

$$\overline{\kappa}(x) = 2\int_0^1 \mathrm{d}u u\kappa(ux) \,, \tag{3.59}$$

then

$$\frac{\mathrm{d}\overline{\kappa}}{\mathrm{d}x} = 2\int_0^1 \mathrm{d}u u^2\kappa'(ux) \leq 0 \tag{3.60}$$

and $\overline{\kappa}(x) \leq \kappa(0) \leq 1$, we see that $\mathrm{d}y/\mathrm{d}x \geq 0$, so that no multiple images can occur.

Images are in odd numbers. A special case is that of singular lenses, i.e. lenses with infinite density at the center: in this case only two images arise when the source is within the radial caustic. This is clear from the discussion in Sect. (2.6): when the singularity is present, the central maximum of the time delay surface is suppressed. Therefore one possible image is missed.

3.2.1 Singular Isothermal Sphere

One of the most widely used axially symmetric model is the Singular Isothermal Sphere (SIS hereafter). The density profile of this model can be derived assuming that the matter content of the lens behaves as an ideal gas confined by a spherically symmetric gravitational potential. This gas is taken to be in thermal and hydrostatic equilibrium. One of the two density profiles satisfying these sets of equations is given by

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \ , \tag{3.61}$$

where σ_v is the velocity dispersion of the "gas" particles and r is the distance from the sphere center. By projecting the three-dimensional density along the line of sight, we obtain the corresponding surface density

$$\begin{aligned}\Sigma(\xi) &= 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{\mathrm{d}z}{\xi^2+z^2} \\ &= \frac{\sigma_v^2}{\pi G}\frac{1}{\xi}\left[\arctan\frac{z}{\xi}\right]_0^\infty \\ &= \frac{\sigma_v^2}{2G\xi} \ . \end{aligned} \tag{3.62}$$

This density profile has a singularity at $\xi = 0$, where the density is ideally infinite. Nevertheless, it has been used to describe the matter distribution in galaxies, expecially because it can reproduce the flat rotation curves of spiral galaxies.

By choosing

$$\xi_0 = 4\pi\left(\frac{\sigma_v}{c}\right)^2 \frac{D_\mathsf{L} D_\mathsf{LS}}{D_\mathsf{S}} \tag{3.63}$$

as the length scale on the lens plane, we obtain:

$$\Sigma(x) = \frac{\sigma_v^2}{2G\xi}\frac{\xi_0}{\xi_0} = \frac{1}{2x}\frac{c^2}{4\pi G}\frac{D_\mathsf{S}}{D_\mathsf{L} D_\mathsf{LS}} = \frac{1}{2x}\Sigma_{\rm cr} \ . \tag{3.64}$$

Thus, the convergence for the singular isothermal profile is

$$\kappa(x) = \frac{1}{2x} \ , \tag{3.65}$$

and the lensing potential (2.22) is

$$\Psi(x) = |x| \ . \tag{3.66}$$

Using Eqs. (2.23), we obtain

$$\alpha(x) = \frac{x}{|x|} \ , \tag{3.67}$$

and the lens equation reads

$$y = x - \frac{x}{|x|} \ . \tag{3.68}$$

If $y < 1$, two solutions of the lens equation exist. They arise at $x = y-1$ and $x = y+1$, on opposite sides of the lens center. The corresponding angular positions of the images are

$$\theta_\pm = \beta \pm \theta_E \tag{3.69}$$

where θ_E is the *Einstein radius*, defined now as

$$\theta_E = \sqrt{\frac{4GM(\theta_E)}{c^2} \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}}} \; . \tag{3.70}$$

The quantity $M(\theta_E)$ is the mass within the Einstein radius. The angular separation between the two images therefore is $\Delta(\theta) = 2\theta_E$: the Einstein radius defines a typical scale for separation between multiple images.

On the other hand, if $y > 1$, Eq. (3.68) has a unique solution, $x = y + 1$. Images arising at $x > 0$ are of type I (positive parity), while those arising at $x < 0$ are of type II (negative parity).

The shear follows from the derivatives of Ψ. Since

$$\frac{\partial \Psi}{\partial x_i} = \frac{x_i}{|x|} \tag{3.71}$$

we have

$$\frac{\partial \Psi}{\partial x_i \partial x_j} = \frac{\delta_{ij} x - x_i x_j / x}{x^2} = \frac{\delta_{ij} x^2 - x_i x_j}{x^3} \; , \tag{3.72}$$

and thus

$$\Psi_{11} = \frac{x^2 - x_1^2}{x^3} = \frac{x_2^2}{x^3} \tag{3.73}$$

$$\Psi_{12} = -\frac{x_1 x_2}{x^3} \tag{3.74}$$

$$\Psi_{22} = \frac{x^2 - x_2^2}{x^3} = \frac{x_1^2}{x^3} \; . \tag{3.75}$$

The shear components are

$$\gamma_1 = \frac{1}{2}(\Psi_{11} - \Psi_{22}) = \frac{1}{2}\frac{x_2^2 - x_1^2}{x^3} = \frac{1}{2}\frac{\sin^2\phi - \cos^2\phi}{x} = -\frac{1}{2}\frac{\cos 2\phi}{x} \; , \tag{3.76}$$

$$\gamma_2 = \Psi_{12} = -\frac{\cos\phi \sin\phi}{x} = -\frac{1}{2}\frac{\sin 2\phi}{x} \; . \tag{3.77}$$

Thus,

$$\gamma(x) = (\gamma_1^2 + \gamma_2^2)^{1/2} = \frac{1}{2x} = \kappa(x) \; . \tag{3.78}$$

From Eq. (3.68), the magnification as a function of the image position is given by

$$\mu = \frac{|x|}{|x| - 1} \; . \tag{3.79}$$

Images are only magnified in the tangential direction, since the radial eigenvalue of the Jacobian matrix is unity everywhere.

If $y < 1$, the magnifications of the two images are

$$\mu_+ = \frac{y+1}{y} = 1 + \frac{1}{y} \quad ; \quad \mu_- = \frac{|y-1|}{|y-1| - 1} = \frac{-y+1}{-y} = 1 - \frac{1}{y} \; , \tag{3.80}$$

from which we see that for $y \to 1$, the second image becomes weaker and weaker until it disappears at $y = 1$. On the other hand, for $y \to \infty$, the source magnification obviously tends to unity: sources which are at large distance from the lens can only be weakly magnified by gravitational lensing.