Introduction to Gravitational Lensing

Lecture scripts

Massimo Meneghetti

Contents

Introduction

These scripts cover most of the topics discussed in my course on Gravitational Lensing.

Their goal is to give an overview on gravitational lensing and on its wide phenomenology. We will start from the basics of the lensing theory, discussing the deflection of light rays and defining some quantities which will be necessary for the rest of the course.

Then, we will discuss lensing on different scales, starting from lensing of point sources by point masses and ending with lensing by large-scale structures on the most extended source on the sky: the Cosmic-Microwave-Background.

Acknowledgements

During the preparation of the lectures, I have been inspired by the lecture notes written by several other authors. In particular several examples have been taken from the Proceedings of the 33rd Saas Fee Advanced Course on Gravitational Lensing, by Kochaneck, Schneider, and Wambsganss. A lot of material was provided by M. Bartelmann, to whom I am very grateful. The very nice tex class used for these scripts was developed by Emanuel Ziegler (ITA).

1 Introduction to lensing

1.1 History of gravitational lensing

1.1.1 Light deflection before GR

Although the gravitational lensing theory has been built in the last century, the hypothesis that light could be deflected by masses is more than 300 years old. It had been speculated even by Newton that masses should deflect light, but he did not know how to describe the deflection properly, because he thought of light as only a wave phenomenon. In 1783, speculating that light consists of corpuscoles, a geologists, astronomer, natural phylosopher and what-so-ever, named John Mitchell (1724-1793) sent to Henry Cavendish (1731-1810) a paper he had written on a method to measure the mass of stars by detecting the reduction in the light speed by effect of gravity as the light corpuscoles propagated from the star's gravitational field to the Earth. Among the other things, in this paper Mitchel suggested that a sufficiently massive body could completely stop the light it emitted and appear as invisible (hey, aren't these balck holes?). This idea was later re-proposed by Pierre-Simon Laplace in 1795. The paper from Mitchell pushed Cavendish to calculate the Newtonian deflection of light for the first time, probably around 1784. Unfortunately, he did not pubilsh his results. Some private notes where discovered only later.

Historical remark: Light deflection in the Newtonian limit

The calculation was as follows (Will, 1988):

- let start from the assumption that light is composed of material corpuscles;
- according to the equivalence principle, the acceleration of a body in a gravitational field is independent of its mass, structure, composition. Therefore we do not need to care about the corpuscle mass;
- any light corpuscle should experience the acceleration

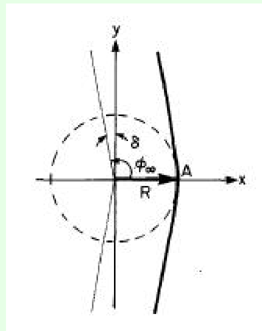
$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{Gm\vec{r}}{r^3}\,, \tag{1.1}$$

  where $\vec{r}$ defines the position of the corpuscle in the gravitational field of the body whose mass is m;

- the solutions of this equation of motion are conic sections. They can describe bound or unbound orbits. However, the speed of light is so large that it exceeds the escape velocity. Thus, the resulting orbit will be an hyperbolic orbit, which can be parametrically written as

$$r = \frac{R(1+e)}{1+e\cos\phi} \quad , \quad r^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = [GmR(1+e)]^{1/2} \,, \tag{1.2}$$

In the previous equations R is the radius of the point of closest approach between the corpuscle and the body of mass m, chosen to lie on the x axis, e is the eccentricity of the orbit and ϕ is an angle, counted from the x axis, called *true anomaly*. r and ϕ define the position of the corpuscle with respect to the mass m in polar coordinates.



- the vecor $\vec{r}$ is written as

$$\vec{r} = r(\vec{e}_x\cos\phi + \vec{e}_y\sin\phi) \tag{1.3}$$

in terms of the two components along the x and the y axes. Thus, the velocity $\vec{v}$ is

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \left(\frac{Gm}{R(1+e)}\right)^{1/2}[-\vec{e}_x\sin\phi + \vec{e}_y(\cos\phi + e)] \,, \tag{1.4}$$

$$v^2 = \frac{Gm}{R(1+e)}(1+2e\cos\phi + e^2) \,. \tag{1.5}$$

- as $r \to \infty$, the trajectory approaches asymptotes that make an angle ϕ_∞ with the x-axis; this occurs when

$$(1+e\cos\phi) = 0 \Rightarrow \cos\phi_\infty = -\frac{1}{e} \,. \tag{1.6}$$

If we define $\phi_\infty \equiv \pi/2 + \delta$, where δ is one-half the deflection angle, then

$$\sin\delta = \frac{1}{e} \,; \tag{1.7}$$

- for determining the deflection angle, we need to determine the eccentricity. Now, let assume that the corpuscle is emitted at infinity with velocity c. Then, from Eq. 1.5 we obtain

$$c^2 = v^2|_{\phi=\phi_\infty} = \frac{Gm}{R(1+e)}(e^2-1) \tag{1.8}$$

$$= \frac{Gm}{R}(e-1) \,. \tag{1.9}$$

Thus,

$$e = \frac{Rc^2}{Gm} + 1 \,; \tag{1.10}$$

- if the massive body is the Sun and the light is grazing its surface,

$$m = M_{\odot} = 1.989 \times 10^{30}\mathrm{kg} \quad (1.11)$$
$$R = R_{\odot} = 6.96 \times 10^{8}\mathrm{m} \quad (1.12)$$

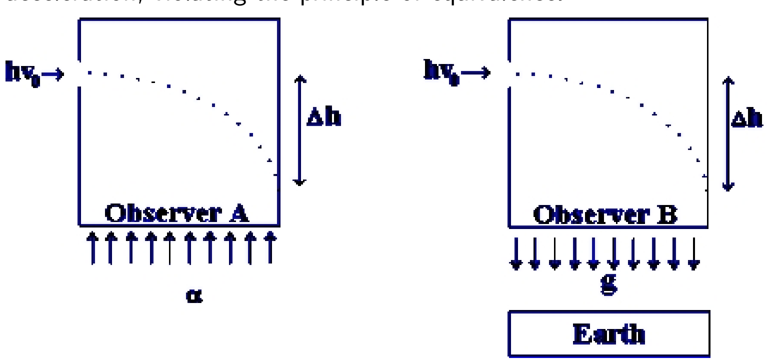
and the deflection angles is

$$\Delta\theta \equiv 2\delta \approx \frac{2Gm}{c^2 R} \approx 0''.875 \quad (1.13)$$

We have to wait until the beginning of the XIXth century for finding an official document by Johann Soldner (1801), where these calculations were published. The result shown above is just one half of the true deflection, because it is derived by neglecting the local curvature of the space-time around massive bodies.

1.1.2 Light deflection in GR

Using an argument based on the principle of equivalence, but still without full equations of relativity, Albert Einstein realized that massive bodies deflect light. The argument works like this. The principle of equivalence states that gravity and acceleration cannot be distinguished. In other words, a free falling observer does not feel gravity and an accelerated observer can interpret the resulting inertial force as due to a gravitational field. Suppose that the observer is contained in a box with a hole on its left side (see upper figure). If the box is accelerated upwards, the observer interprets the inertial force on him as a gravitational force acting downwards. Suppose that a light ray enters the hole on the left side of the box and propagates towards right. As the box is moving upwards, the ray hits the wall of the box on the opposite side at a lower point than it enter. As the box is accelerated the light ray appears curved. Then, based on the principle of equivalence, light must be deflected by gravity. Indeed, we can imagine to reverse the experiment: let the box to be stationary and within the gravitational field whose intesity is such to resemble the previous acceleration. If light is not deflected by gravity, then the oberver has the possibility to discriminate between gravity and acceleration, violating the principle of equivalence.



In order to get the correct value of the deflection of light by a mass M, we need to use the Theory of General Relativity (Einstein, 1916). According to this theory, the deflection is described by geodesic lines following the curvature of the space-time. In curved space-time, geodesic lines are lines which are as "straight as possible", resembling straight lines in flat space-time. As a light ray follows the curvature, it is bent towards the mass which causes the space-time to be curved. This bending gives rise to several important phenomena:

- multiple paths around a single mass become possible, e.g. one around the left and one around the right side of the deflector. The observer, who will see an image of the source along the backward tangent of each ray arriving at his position, will then see multiple images of a single source;

- in addition, the light deflection of two neighbouring rays may be different. Suppose a pair of rays, one from one side and one from the other side of a source, passes by a lensing mass distribution. The ray which passes closer to the deflector will be bent more than the other, thus the source will appear stretched. It is thus expected that gravitational lensing will typically distort the sources. By the same mechanism, they can appear larger or smaller than they originally are;

- since photons are not created, neither destroyed by the lensing effect the surface brightness of the source will remain unchanged. Since, as we said, the size is not conserved, this implys that the source can be either *magnified* or *demagnified* by lensing. If it is enlarged it will appear brigher, otherwise fainter;

- in case that multiple light paths are possible between the source and the observer, since they will be characterized by different lengths, the light travel times will differ for the different images. One of the images will appear first, the others will be delayed.

Starting from the equivalence principle, we are thus arrived at the expectation of multiple images, distortions, magnification, and time delayes. All of these phenomena have been observed in numerous cases.

In 1919, Lodge used for the first time the term 'lens' in the context of gravitational light deflection. In 1924, Chowlson studied the case of a source perfectly aligned with a foreground mass, concluding that the source should be imaged as a ring around the lens. These rings are nowadays known as 'Einstein Rings'. In fact, this idea had been already developed by Einstein before, as it has been established from some unpublished notes by the scientist dated 1912.

For several years, the above mentioned effects of gravitational lensing were belived to be unobservable. For example, in 1936 Einstein, after being approached by a Czech engineer, Rudi Mandl, calculated the properties of a lensing star, including the image positions, their separations, and their magnifications. He concluded that the angular separation between the multiple images of a background star was way too small for being detectable (of order milliarcseconds).

This pessimistic view was not shared by Fritz Zwicky who in 1937 firstly considered the potential lensing effects of "extragalactic nebulae" (Zwicky, 1937c). Given his estimates of the nebulae masses (Zwicky, 1937b), he argued that the typical image separations in these cases should be of order 10' arcseconds, thus observable. He also discuss a method to derive the nebulae masses through lensing. He estimated that about one out of 400 distant sources should be affected by lensing (Zwicky, 1937a).

1.1.3 Lensing in the '60s

Due to the lack of suitable instrumentation for this kind of studies, lensing made little progress until the '60s. During 1963/64 three papers gave new impulse to the field:

- Klimov (1963) considered lensing of galaxies by galaxies, concluding that, for sufficient alignment, a ring-shaped image would occur and could be easily detectable, whereas if the alignment were imperfect, multiple galaxy images would appear, which would be difficult to distinguish from double or multiple galaxies;

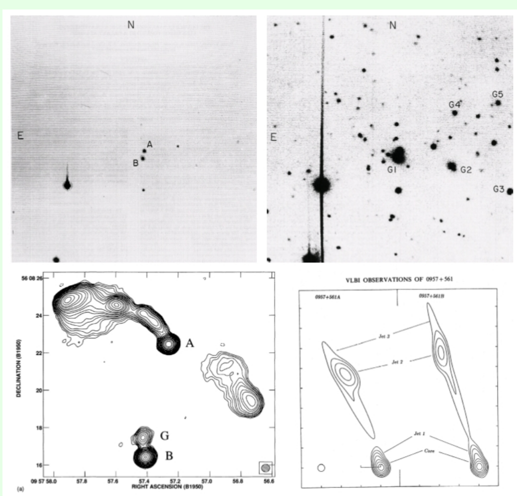
- Liebes (1964) studied lensing of stars on stars, of stars on globular clusters in our galaxy, and of stars on stars if both are members of the same globular cluster. He also considered the possibility that stars in our galaxy can lens stars in M31;

- Refsdal (1964) extensively studied the properties of point lenses and considered the time-delay between the two images, due to the different light-travel-time along light rays corresponding to each image. In particular, he argued that geometrical optics can be used safely in considering gravitational lensing effects. He pointed out that the time delay depends on the mass of the lens and on the distances to the lens and the source, and concluded that, if the image separation and the time delay could be measured, the lens mass and the Hubble constant could be determined.

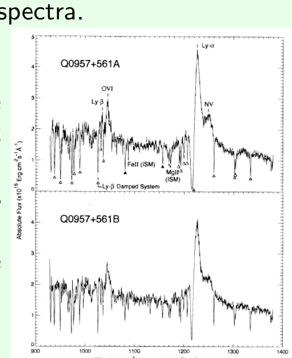
At the same time, in 1963, the first quasars were detected, i.e. a new population of compact and luminous sources far behind the Zwicky's nebulae. After that, it was just a matter of time before the first case of extragalactic lensing event was observed. This happened about 15 years later.

Historical remark: The first detection of a double quasar: Q0957+561

The first case of multiple imaging was discovered by Walsh et al. (1979). During a campaign for optically identifying radio sources, they found a pair of quasars separated by about 6 arcseconds (upper left panel in the Fig. below, where the two quasars are indicated by the letters A and B), having identical colors, redshifts ($z_{\mathrm{s}} = 1.41$), and spectra. The system was named Q0957+561.

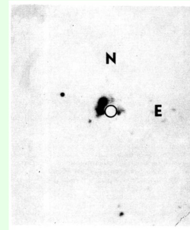


In that year the first CCD cameras substituted the photometric plates and the Very-Large-Array radio interferometer started to operate. In a deeper optical observation, a galaxy was detected between the two quasars (Stockton, 1980; Young et al., 1980, ; see upper right panel). The galaxy was idetified as a member of a small cluster of galaxies at $z_{\mathrm{d}} = 0.36$. The lower left panel in the Figure shows the 6cm VLA map of the system (Harvanek et al., 1997). The two quasars are both compact radio sources with similar radio spectra.

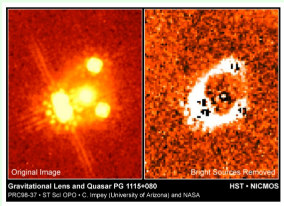


The milli-arcsecond structure of the two compact components A and B, as observed with the VLBI, is shown in the lower-right panel (Gorenstein et al., 1988), where it is seen that the two components have both a core-jet structure, that one jet is a linearly transformed version of the other, and they are mirror symmetric. This is predicted by any generic lens model which assigns opposite parity to the images. The figure on the left side shows the optical spectra of the two images, which exhibit the same features. This is a further confirmation of the lensing nature of the double quasar.

Historical remark: The "triple" quasar PG1115+080



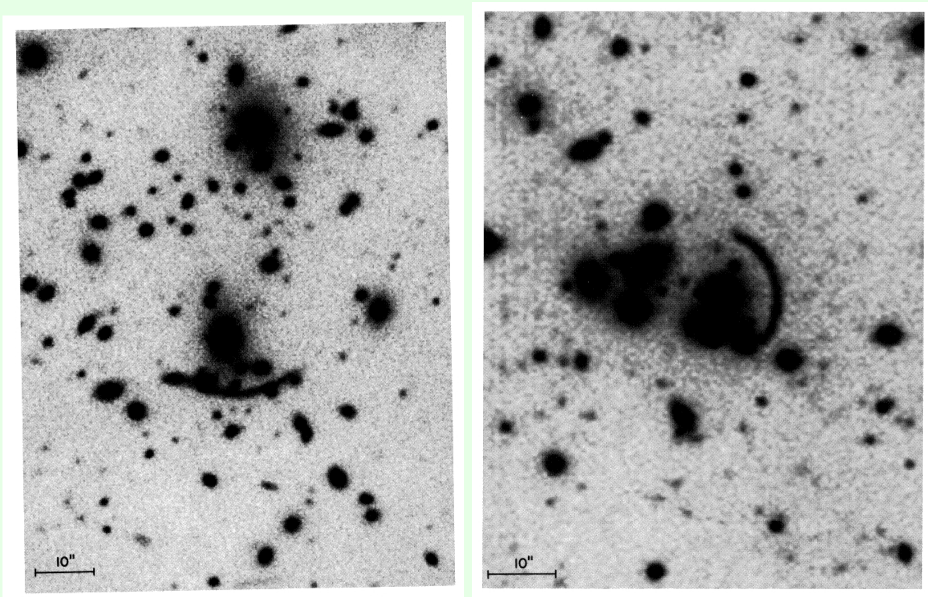
About one year later than the discovery of Q0957+561, Weymann et al. (1980) reported the discovery of another lens candidate: PG1115+080. In the photograph on the right, they identified three potential images of a QSO at $z_{\rm s} = 1.72$, of which one was much brighter than the others. Later, this image was recognized to be a blend of two images separated by $0.5''$. The lens galaxy is at redshift $z_{\rm d} = 0.31$.



This is much clearer in the image shown on the left. In the left panel we see an IR observation carried out by HST. Note that when the light from the bright sources is subtracted a clear ring-like structure appears, as shown in the right panel. This is the image of the host galaxy of the QSO, distorted by the lens galaxy, and mapped into a nearly complete Einstein ring.

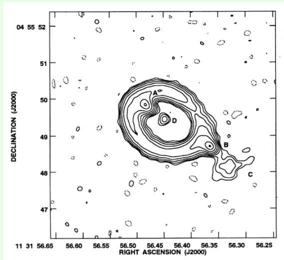
Historical remark: The first gravitational arcs

The first detection of gravitational arcs in galaxy clusters is dated 1986. In this year, two groups independently discovered strongly elongated, curved features around two clusters of galaxies (Soucail et al., 1987; Lynds & Petrosian, 1989): A370 (left panel below) and CL2244-02 (right panel).

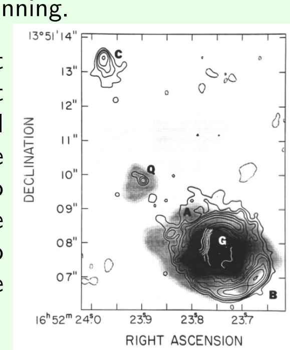


They were seen displaced from the cluster center and curving around it. Several hypothesis were put forward about the nature of these features, all proven wrong. The correct interpretation of these observations as gravitational lensing effects was made by Paczynski (1987), when the redshift of the arc in A370 was measured and discovered to be much larger than the redshift on the cluster. In particular, A370 is at redshift $z_{\rm d} = 0.374$, while the arc is at redshift $z_{\rm s} = 0.724$. The arc in CL2244-02 ($z_{\rm d} = 0.3$) is at redshift $z_{\rm s} = 2.24$. The figures below show color higher quality images of the same clusters observed with HST and with ISAAC@VLT.

Historical remark: The first Einstein rings



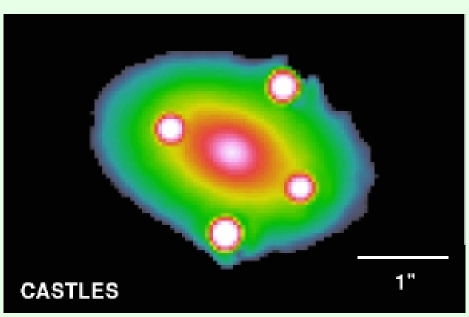
The first Einstein ring was discovered by (Hewitt et al., 1988). It consists of a radio ring, image of a QSO known as MG1131+0456. The discovery of such lensing feature came as a surprise, since rings were expected only for axially symmetric lenses, and previous multiply imaged QSOs were suggesting that galaxies were far from being axially symmetric. Due to the faint optical counterpart of the ring, the lensing nature of this system remained unclear at the beginning.

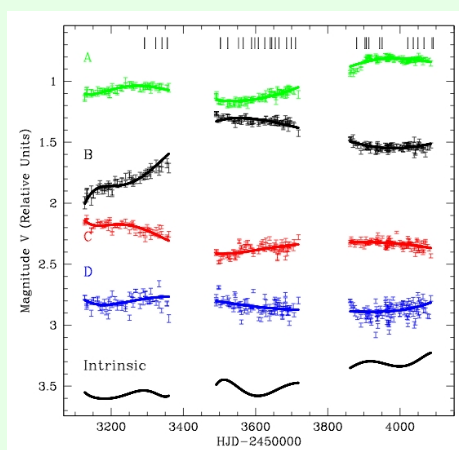


The first easily confirmed case of Einstein ring is that of the quasar MG1654+13 (Langston et al., 1989). It is shown in the Figure on the right both as an optical image (gray scale) and in the radio (contours). The optical QSO is labelled Q. In the radio it exhibits two radio lobes, with emissivity peaks at A, B, and C. The southern lobe is lensed by the galaxy G and mapped into an Einstein ring. The peak A is a counter image of the peak B.

Historical remark: Quasar microlensing

As we have seen, distant quasars can be multiply imaged by foreground galaxies. Galaxies don't have smooth mass distributions. At least part of them is made of stars, which act themselves as microlenses. These microlenses can split the macro-images of quasars into many micro-images. The typical angular separation of the micro-images is of order of few microarcseconds, thus they unresolvable. However, as we will see in more detail in the microlensing section, the magnification of the quasar images can be changed by the micro-lenses. Since the stars move with respect to the background quasar, the magnification pattern varies with time. Thus, if the quasar has multiple images, and each of them is affected by microlensing, such effect could be revealed by uncorrelated variations in the light curves. This effect, which was predicted by several authors (Chang & Refsdal, 1979, 1984; Paczynski, 1986b; Kayser et al., 1986; Schneider & Weiss, 1987), was detected in the four images quasar lens QSO 2237+0305 (Irwin et al., 1989).

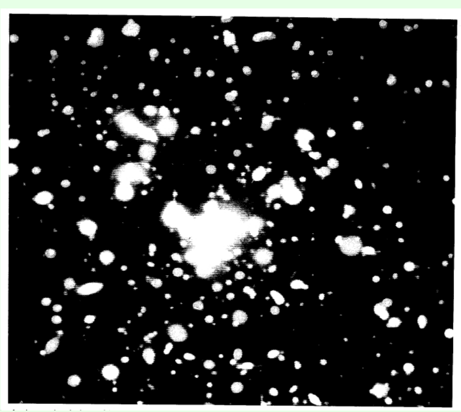


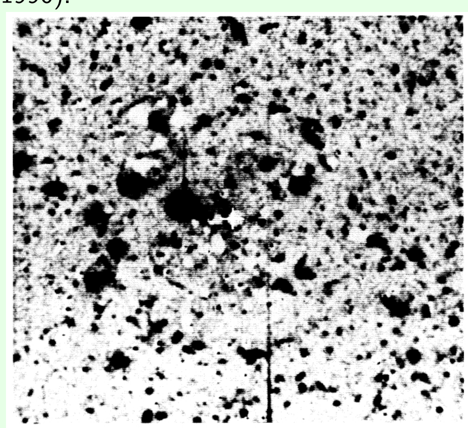


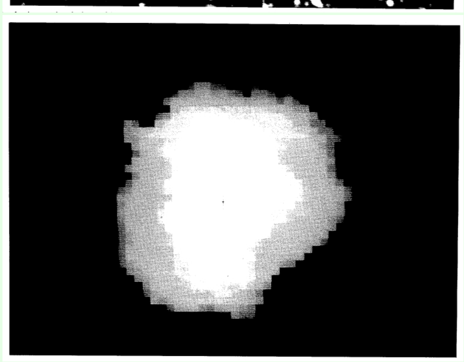
Credits: CASTLES (left image), Eigenbrod et al. (2008) (right image)

Historical remark: First weak lensing measurents

Multiple images, giant arcs, etc. are examples of strong lensing features. These features are rather rare because, as we will see, they can show up only under very particular conditions. In particular, the lens and the source must be well aligned along the line-of-sight. If the source lays at relatively large angular distances from the centre of the lens, its light will feel the gravitational effect of the lens, but with much smaller intensity. Let think to a galaxy cluster lensing background galaxies. While at the center of these structures we may find giant arcs or arclets, going towards the outskirts the shape of the sources will become more and more similar to their intrinsic shape. Thus it will become more and more difficult to detect the lensing effect of the cluster by looking at a single galaxy, which is intrisically elliptical, i.e. elongated in some direction. On the other hand, lensing distorts the shape of the sources tangentially with respect to the lens center (in fact, we will see that also radial elongations are possible in the strong lensing regime). Thus, close ensembles of background galaxies are distorted coherently in the tangential direction. Such coherent alignment can be detected by averaging the ellipticity vectors over several nearby galaxies: if galaxies have random orientations, their mean intrinsic ellipticity will vanish and what will remain is the ellipticity induced by lensing. The first clusters where the weak lensing effect was measured and used to determine the shape of the lens clusters were A1689 and CL1409 (Tyson et al., 1990). The figures below show the analysis carried out on A1689. In the first, we see the color image of the cluster. The second panel shows the same image after subtracting the R-band observation, in order to remove part of the light from the cluster galaxies. Several arclets appear in the background of the cluster and some coherent tangential alignment between the images can be recognized. The last figure shows a map of the mean tangential alignment, which can be used to infer the shape of the cluster (from Tyson et al 1990).



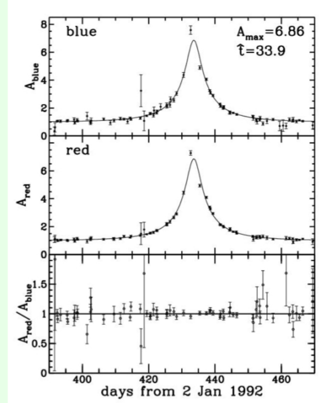




Historical remark: Galactic microlensing

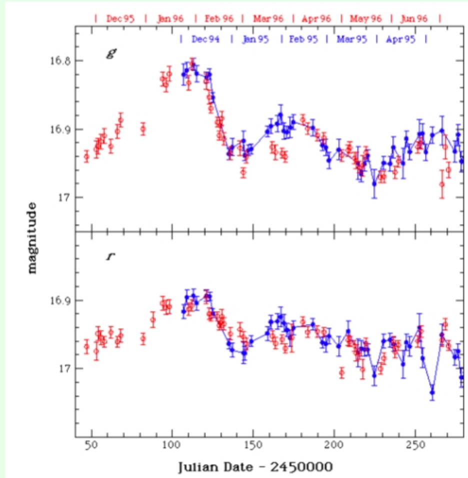
In 1986, B. Paczyński proposed to monitor the stars of the Large-Magellanic-Cloud (LMC) searching for variations of their light curves due to microlensing effects. The idea behind this was to check wether the dark matter necessary to explain the flat rotation curve of our galaxy (and of other spiral galaxies) is in form of compact objects such as brown dwarfs, neutron stars, Jupiter-like planets, black holes, etc. Such objects would produce microlensing events, magnifying the the stars in the LMC. Since the potential microlenses in the halo of the galaxy are in relative motion with respect to the source stars, the microlensing would show up as flux variations in the light curves. Paczyński estimated that only 1 out of 10^7 stars would have shown microlensing features at any time, so that since the beginning it was clear that such a monitoring campaign would have involved world-wide collaborations and several observatories. Two large groups started the measure the light curves of the LMC and SMC stars at the beginning of the '90s. The first microlensing event towards the LMC was discovered in 1993 by two independent groups (Alcock et al., 1993; Aubourg et al., 1993, for the MACHO and the EROS collaborations). In the same year another group announced the discovery of a microlensing event towards the galactic bulge (Udalski et al., 1993, ; OGLE collaboration).

The light curves in the B and in the R bands of the first galactic microlensing event MACHO-LMC-1 are shown in the Fig. on the side. The error bars indicate the data, while the solid line is the best fit with a standard single-lens model. As we can see, one data point near the maximum of the curve is significantly off. This may indicate that an additional lens component is necessary for fully explaining the observation (binary microlensing?). Note that the same effect is seen in the two bands: since lensing is achromatic, this is another proof of the lensing nature of the flux variation observed in the light curves of this LMC star.



Historical remark: Time delays

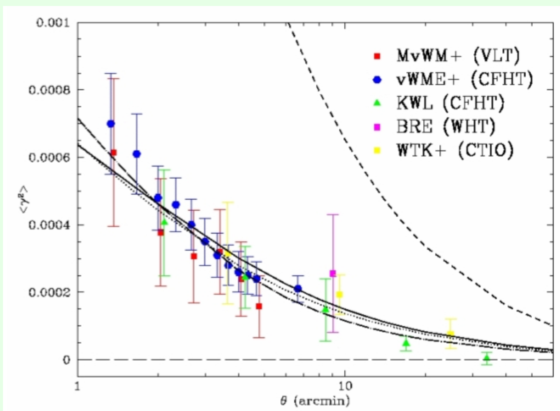
Quasars are intrinsically variable sources. Following the Refsdal's idea of measuring the Hubble constant from the time delays between the multiple images of the same source, the lighet curves of the double QSO 0957+561 were monitored by several groups both in the optical and in the radio bands (see e.g. Vanderriest et al., 1989; Schild, 1990; Lehar et al., 1992). From the light-curves, estimates of the time delay were derived which were significantly different. The major uncertainties were due to seasonal gaps between observations and to the uncorrelated variability caused by microlensing in the lens galaxy.



To account for these effects several methods were developed, yielding different results, with time delays ranging between 410 and 540 days. The issue was put to rest when a relatively sharp variation of the flux of the leading image was detected in December 1994 (see Fig. Kundic et al., 1995). Each of the two estimates for the time delay predicted a different epoch for the occurrence of the corresponding feature in the other image. This feature was observed in the trailing image in February 1996 (Kundic et al., 1997). In the Fig., the light-curves of the images A and B of the QSO are shown in two filters. They have been shifted in time and flux, such to match the flux variations observed in the two images. The time shift corresponds to 417 ± 3 days, thus the controversy was resolved in favor of the short delay.

Historical remark: Cosmic shear

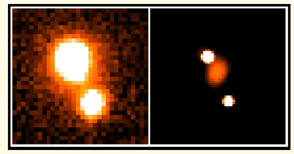
The theory of cosmic shear was developed in the early '90s (Blandford et al., 1991; Miralda-Escude, 1991; Kaiser, 1992). The basic idea behind that is that the universe itself is a huge gravitational lens, and that the shape of the distant galaxies which we can measure is changed by the lensing effects by any structure along the line of sight. By observing large fields and averaging over huge samples of galaxies, we can study how the galaxy distortions, or cosmic shear, are correlated on different scales. Such a measurement contains information about the distribution of matter in universe, the matter power-spectrum, and the growth of the cosmic structures. This is nowadays believed to be one of the most powerful methods to investigate the dark energy component of the universe. The first detections of cosmic shear were announced by four independent groups only in 2000 (Bacon et al., 2000; Kaiser et al., 2000; Van Waerbeke et al., 2000; Wittman et al., 2000), after the development of the wide-field CCD mosaic cameras made it possible to observe the large fields necessary of this kind of observations.



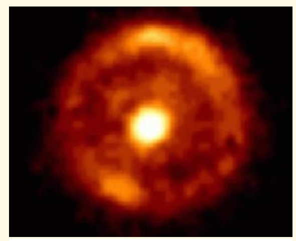
The Fig. on the side shows the first cosmic shear measurements based on $\sim 10^5$ galaxies covering ~ 1 sq. degree on the sky. It shows the shear dispersion as a function of equivalent circular aperture radius as obtained by the groups mentioned above. Shown are also the results obtained by Maoli et al. (2001) about a year later by analyzing 50 uncorrelated VLT fields. The curves show the prediction of the shear signal in several cosmological models.

Example: Multiply-imaged quasars

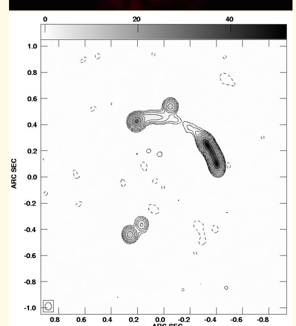
Identification of the lensing galaxy in a double quasar system: the left panel shows on infrared (J-band) observation of the two images of double quasar HE 1104-1825 ($z_Q = 2.316$, $\theta = 3.2''$). The right panel obtained with some new deconvolution technique nicely reveals the lensing galaxy (at $z_G = 1.66$) between the quasar images (Credits: European Southern Observatory).



Example: Einstein ring



B1938+666 is another multiple-image lens, and was discovered in JVAS (Jodrell/VLA Astrometric Survey). This is a survey of flat-spectrum radio sources designed to identify gravitational lens candidates. HST observations show an Einstein ring in IR. The lens redshift is 0.878, but the source redshift is not yet known (IR spectroscopy required).

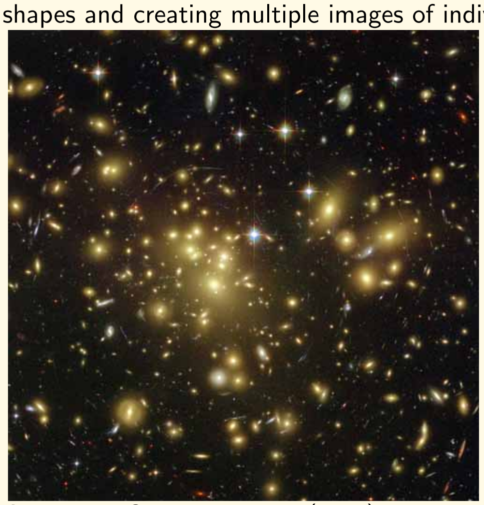


The bottom figure shows a MERLIN image of this system at 5GHz. In radio there is a significant arc visible.

Credit: JVAS/CLASS

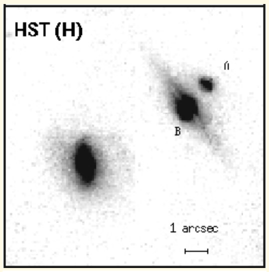
Example: Arcs in galaxy clusters

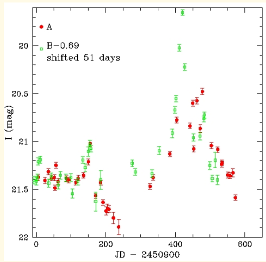
Abell 1689 is a galaxy cluster at z=0.183. The gravity of the cluster's trillion stars - plus dark matter - acts as a 2-million-light-year-wide 'lens' in space. This 'gravitational lens' bends and magnifies the light of galaxies located far behind it, distorting their shapes and creating multiple images of individual galaxies.



Credit: NASA, N. Benitez (JHU), T. Broadhurst (The Hebrew University), H. Ford (JHU), M. Clampin(STScI), G. Hartig (STScI), G. Illingworth (UCO/Lick Observatory), the ACS Science Team and ESA.

Example: Time delays





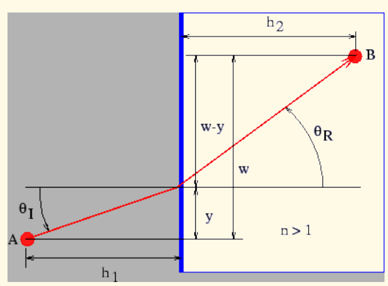
B1600+434 is a double gravitational lens system. A distant QSO at redshift $z = 1.59$ is lensed by an edge-on-late-type galaxy at $z = 0.41$ and has two images, labeled with A and B in the upper image. QSO's are characterized by intrinsic variability of their luminosity. The light curves of the two images have the same shape, as expected since they arise from the same source. However, the light curve of the image B is shifted by ~ 50 days with respect to that of image A. The reason is the different path of the light coming from the two images.
Credit: I. Burud, Institut d'Astrophysique et de Gophysique de Lige, Avenue de Cointe 5, B-4000 Lige, Belgium

1.2 Fermat's principle and light deflection

Starting from the field equations of general relativity, light deflection can be calculated by studying geodesic curves. It turns out that light deflection can equivalently be described by Fermat's principle, as in geometrical optics. This will be our starting point.

Example: Fermat's Principle in geometrical optics

In its simplest form the Fermat's principle says that light waves of a given frequency traverse the path between two points which takes the least time. The speed of light in a medium with refractive index n is c/n, where c is its speed in a vacuum. Thus, the time required for light to go some distance in such a medium is n times the time light takes to go the same distance in a vacuum.



Referring to the figure above, the time required for light to go from A to B becomes

$$t = [\{h_1^2 + y^2\}^{1/2} + n\{h_2^2 + (w-y)^2\}^{1/2}]/c.$$

We find the minimum time by differentiating t with respect to y and setting the result to zero, with the result that

$$\frac{y}{\{h_1^2 + y^2\}^{1/2}} = n\frac{w-y}{\{h_2^2 + (w-y)^2\}^{1/2}}.$$

However, we note that the left side of this equation is simply $\sin\theta_I$, while the right side is $n\sin\theta_R$, so that the minimum time condition reduces to

$$\sin\theta_I = n\sin\theta_R$$

We recognize this result as Snell's law.

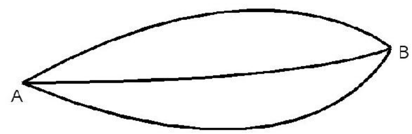
We first need an index of refraction n because Fermat's principle says that light will follow a path along which the travel time,

$$\int \frac{n}{c} \mathrm{d}l \,, \tag{1.14}$$

will be extremal. As in geometrical optics, we thus search for a path, $\vec{x}(l)$, for which the variation

$$\delta \int_A^B n(\vec{x}(l)) \mathrm{d}l = 0 \,, \tag{1.15}$$

where the starting point A and the end point B are kept fixed.



In order to find the index of refraction, we make a first approximation: we assume that the lens is weak, and that it is small compared to the overall dimensions of the optical system composed of source, lens and observer. With "weak lens", we mean a lens whose Newtonian gravitational potential Φ is much smaller than c^2, $\Phi/c^2 \ll 1$. Note that this approximation is valid in virtually all cases of astrophysical interest. Consider for istance a galaxy cluster: its gravitational potential is $|\Phi| < 10^{-4}c^2 \ll c^2$.

The metric of unperturbed space-time is the Minkowski metric,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,,$$

whose line element is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (\mathrm{d}x^0)^2 - (\mathrm{d}\vec{x})^2 = c^2\mathrm{d}t^2 - (\mathrm{d}\vec{x})^2 \,. \tag{1.16}$$

A weak lens perturbs this metric such that

$$\eta_{\mu\nu} \to g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1-\frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1-\frac{2\Phi}{c^2}) & \\ 0 & 0 & 0 & -(1-\frac{2\Phi}{c^2}) \end{pmatrix}$$

for which the line element becomes

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2\Phi}{c^2}\right)(\mathrm{d}\vec{x})^2 \,. \tag{1.17}$$

Now light propagates at zero eigentime, $\mathrm{d}s = 0$, from which we gain

$$\left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 = \left(1-\frac{2\Phi}{c^2}\right)(\mathrm{d}\vec{x})^2 \,. \tag{1.18}$$

The light speed in the gravitational field is thus

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}} \approx c\left(1+\frac{2\Phi}{c^2}\right) \,, \tag{1.19}$$

where we have used that $\Phi/c^2 \ll 1$ by assumption. The index of refraction is thus

$$n = c/c' = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2} \; . \tag{1.20}$$

With $\Phi \leq 0$, $n \geq 1$, and the light speed c' is lower than in vacuum.
n will typically depend on the spatial coordinate $\vec{x}$ and perhaps also on time t. Let $\vec{x}(l)$ be a light path. Then the light travel time is proportional to

$$\int_A^B n[\vec{x}(l)]\mathrm{d}l \; , \tag{1.21}$$

and the light path follows from

$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0 \; . \tag{1.22}$$

This is a standard variational problem, which leads to the well known Euler equations. In our case we write

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda \; , \tag{1.23}$$

with a curve parameter λ which is yet arbitrary, and find

$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0 \tag{1.24}$$

The expression

$$n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \equiv L(\dot{\vec{x}}, \vec{x}, \lambda) \tag{1.25}$$

takes the role of the Lagrangian in analytic mechanics, with

$$\dot{\vec{x}} \equiv \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} \; . \tag{1.26}$$

Finally, we have

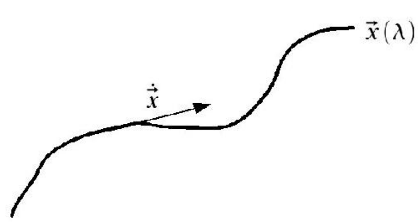
$$\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = |\dot{\vec{x}}| = (\dot{\vec{x}}^2)^{1/2} \; . \tag{1.27}$$

Using these expressions, we find the Euler equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0 \; . \tag{1.28}$$

Now,

$$\frac{\partial L}{\partial \vec{x}} = |\dot{\vec{x}}|\frac{\partial n}{\partial \vec{x}} = (\vec{\nabla} n)|\dot{\vec{x}}| \; , \; \frac{\partial L}{\partial \dot{\vec{x}}} = n\frac{\dot{\vec{x}}}{|\dot{\vec{x}}|} \; . \tag{1.29}$$



Evidently, $\dot{\vec{x}}$ is a tangent vector to the light path , which we can assume to be normalized by a suitable choice for the curve parameter λ. We thus assume $|\dot{\vec{x}}| = 1$ and write $\vec{e} \equiv \dot{\vec{x}}$ for the unit tangent vector to the light path. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(n\vec{e}) - \vec{\nabla} n = 0 \; , \tag{1.30}$$

or

$$n\dot{\vec{e}} + \vec{e} \cdot [(\vec{\nabla} n)\dot{\vec{x}}] = \vec{\nabla} n \; , \tag{1.31}$$

$$\Rightarrow n\dot{\vec{e}} = \vec{\nabla} n - \vec{e}(\vec{\nabla} n \cdot \vec{e}) \; . \tag{1.32}$$

The second term on the right hand side is the derivative along the light path, thus the whole right hand side is the gradient of n perpendicular to the light path. Thus

$$\dot{\vec{e}} = \frac{1}{n}\vec{\nabla}_\perp n = \vec{\nabla}_\perp \ln n \; . \tag{1.33}$$

As $n = 1 - 2\Phi/c^2$ and $\Phi/c^2 \ll 1$, $\ln n \approx -2\Phi/c^2$, and

$$\dot{\vec{e}} \approx -\frac{2}{c^2}\vec{\nabla}_\perp \Phi \; . \tag{1.34}$$

The total deflection angle of the light path is now the integral over $-\dot{\vec{e}}$ along the light path,

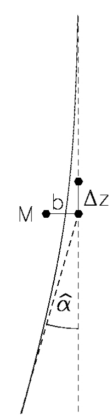
$$\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \mathrm{d}\lambda \; . \tag{1.35}$$

The deflection is thus the integral over the "pull" of the gravitational potential perpendicular to the light path. Note that $\vec{\nabla}\Phi$ points away from the lens centre, so $\hat{\vec{\alpha}}$ points in the same direction.

As it stands, the equation for $\hat{\vec{\alpha}}$ is not useful, as we would have to integrate over the actual light path. However, since $\Phi/c^2 \ll 1$, we expect the deflection angle to be small. Then, we can adopt the Born approximation familiar from scattering theory and integrate over the unperturbed light path.

Suppose, therefore, a light ray starts out into $+\vec{e}_z$-direction and passes a lens at $z = 0$, with impact parameter b. The deflection angle is then given by

$$\hat{\vec{\alpha}}(b) = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi \mathrm{d}z \tag{1.36}$$



Special case: point mass lens

If the lens is a point mass, then

$$\Phi = -\frac{GM}{r} \tag{1.37}$$

with $r = \sqrt{x^2+y^2+z^2} = \sqrt{b^2+z^2}$, $b = \sqrt{x^2+y^2}$ and

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \frac{GM}{r^3}\begin{pmatrix} x \\ y \end{pmatrix} . \tag{1.38}$$

The deflection angle is then

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2}\begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{(b^2+z^2)^{3/2}} \\ &= \frac{4GM}{c^2}\begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2(b^2+z^2)^{1/2}}\right]_0^\infty = \frac{4GM}{c^2 b}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} , \end{aligned} \tag{1.39}$$

with

$$\begin{pmatrix} x \\ y \end{pmatrix} = b\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{1.40}$$

Notice that $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius of a (point) mass M, thus

$$|\hat{\vec{\alpha}}| = \frac{4GM}{c^2 b} = 2\frac{R_s}{b} . \tag{1.41}$$

Also notice that $\hat{\vec{\alpha}}$ is linear in M, thus the deflection angles of an array of lenses can linearly be superposed.
Note that the deflection angle found here in the framework of general relativity exceeds by a factor of two that calculated by using standard Newtonian Gravity (see Eq. 1.13), as anticipated at the beginning of this chapter.

Since the speed of the light is reduced in the gravitational field, $c' = c/n$, the travel time (along the perturbed path) is larger by

$$\Delta t = \int \frac{\mathrm{d}l}{c'} - \int \frac{\mathrm{d}l}{c} = \frac{1}{c}\int (n-1)\mathrm{d}l = -\frac{2}{c^3}\int \Phi \mathrm{d}l . \tag{1.42}$$

This is the so-called *Shapiro delay* (Shapiro, 1964).

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