2 General concepts

2.1 The general lens

The deflection angle in Eq. 1.41 depends linearly on the mass M. This results was obtained by linearizing the equations of general relativity in the weak field limit. Under these circumstances, the superposition principle holds and t the deflection angle of an array of lenses can be calculated as the sum of all contributions by each single lens. Suppose we have a sparse distribution of N point masses on a plane, whose positions and masses are $\vec{\xi}_i$ and M_i, $1 \leq i \leq N$. The deflection angle of a light ray crossing the plane at $\vec{\xi}$ will be:

$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2} \ . \tag{2.1}$$

We now consider more realistic lens models, i.e. three dimensional distributions of matter. Even in the case of lensing by galaxy clusters, the physical size of the lens is generally much smaller than the distances between observer, lens and source. The deflection therefore arises along a very short section of the light path. This justifies the usage of the *thin screen approximation* (see Fig. (2.1)): the lens is approximated by a planar distribution of matter, the lens plane. Even the sources are assumed to lie on a plane, called the source plane.

Within this approximation, the lensing matter distribution is fully described by its surface density,

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\ \mathrm{d}z \ , \tag{2.2}$$

where $\vec{\xi}$ is a two-dimensional vector on the lens plane and ρ is the three-dimensional density.

As long as the thin screen approximation holds, the total deflection angle is obtained by summing the contribution of all the mass elements $\Sigma(\vec{\xi})\mathrm{d}^2\xi$:

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}\ \mathrm{d}^2\xi' \ . \tag{2.3}$$

2.2 Lens equation

In Fig. (2.1) we sketch a typical gravitational lens system. A mass concentration is placed at redshift z_L, corresponding to an angular diameter distance D_L. This lens deflects the light rays coming from a source at redshift z_S (or angular distance D_S).

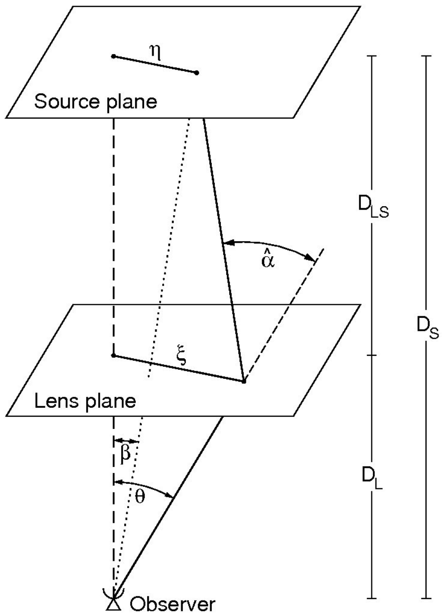


Figure 2.1: Sketch of a typical gravitational lensing system (Figure from Bartelmann & Schneider, 2001).

Remark:
It is not guaranteed that the relation between physical size, distance and angular size can be written as $[\text{physical size}] = [\text{angular size}] \cdot [\text{distance}]$ if space is curved. It is however possible to define distances in curved spacetime such that this relation from Euclidean space holds. Note, however, that due to cosmic expansion, distances are not additive, such that $D_\mathsf{L} + D_\mathsf{LS} \neq D_\mathsf{S}$.

We first define an optical axis, indicated by the dashed line, perpendicular to the lens and source planes and passing through the observer. Then we measure the angular positions on the lens and on the source planes with respect to this reference direction.

Consider a source at the angular position $\vec{\beta}$, which lies on the source plane at a distance $\vec{\eta} = \vec{\beta} D_\mathsf{S}$ from the optical axis. The deflection angle $\hat{\vec{\alpha}}$ of the light ray coming from that source and having an impact parameter $\vec{\xi} = \vec{\theta} D_\mathsf{L}$ on the lens plane is given by Eq. (1.36). Due to the deflection, the observer receives the light coming from the source as if it was emitted at the angular position $\vec{\theta}$.

If $\vec{\theta}$, $\vec{\beta}$ and $\hat{\vec{\alpha}}$ are small, the true position of the source and its observed position on the sky are related by a very simple relation, obtained by a geometrical construction. This relation is called the *lens equation* and is written as

$$\vec{\theta} D_\mathsf{S} = \vec{\beta} D_\mathsf{S} + \hat{\vec{\alpha}} D_\mathsf{LS} \ , \tag{2.4}$$

where D_LS is the angular diameter distance between lens and source.

Defining the reduced deflection angle

$$\vec{\alpha}(\vec{\theta}) \equiv \frac{D_\mathsf{LS}}{D_\mathsf{S}} \hat{\vec{\alpha}}(\vec{\theta}) \ , \tag{2.5}$$

from Eq. (2.4), we obtain

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \ . \tag{2.6}$$

This equation, called *lens equation* is apparently very simple. All the interesting physics of lensing arises because $\vec{\alpha}$ depends on $\vec{\theta}$.

It is very common and useful to write Eq. (2.4) in dimensionless form. This can be done by defining a length scale ξ_0 on the lens plane and a corresponding length scale $\eta_0 = \xi_0 D_S/D_L$ on the source plane. Then we define the dimensionless vectors

$$\vec{x} \equiv \frac{\vec{\xi}}{\xi_0} \quad ; \quad \vec{y} \equiv \frac{\vec{\eta}}{\eta_0} \ , \tag{2.7}$$

as well as the scaled deflection angle

$$\vec{\alpha}(\vec{x}) = \frac{D_L D_{LS}}{\xi_0 D_S} \hat{\vec{\alpha}}(\xi_0 \vec{x}) \ . \tag{2.8}$$

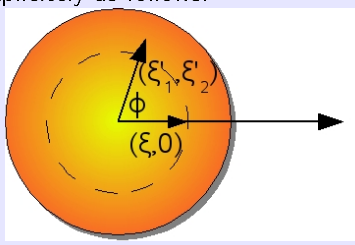
Carrying out some substitutions, Eq. (2.4) can finally be written as

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \ . \tag{2.9}$$

Special case: axially symmetric lenses

In general, the deflection angle is a two-dimensional vector. In the case of axially symmetric lenses we may compute it only in one dimension, since all light rays from the source to the observer must lie in the plane spanned by the center of the lens, the source and the observer. This can be seen explicitely as follows.

We start from Eq. 2.3. Let take the lens center as the origin of the reference frame. By symmetry, we can choose the reference frame such that $\vec{\xi} = (\xi, 0)$, $\xi \geq 0$. In polar coordinates, $\vec{\xi'} = (\xi'_1, \xi'_2) = \xi'(\cos\phi, \sin\phi)$.



Then,

$$\begin{aligned} \vec{\xi} - \vec{\xi'} &= (\xi - \xi'\cos\phi, -\xi'\sin\phi) & (2.10) \\ |\vec{\xi} - \vec{\xi'}|^2 &= \xi^2 + \xi'^2\cos^2\phi - 2\xi\xi'\cos\phi + \xi'^2\sin^2\phi \\ &= \xi^2 + \xi'^2 - 2\xi\xi'\cos\phi & (2.11) \end{aligned}$$

For a symmetric mass distribution $\Sigma(\vec{\xi}) = \Sigma(|\vec{\xi}|)$. The components of the deflection angle are thus

$$\begin{aligned} \hat{\alpha}_1(\vec{\xi}) &= \frac{4G}{c^2}\int_0^\infty d\xi'\xi'\Sigma(\xi')\int_0^{2\pi} d\phi \frac{\xi - \xi'\cos\phi}{\xi^2 + \xi'^2 - 2\xi\xi'\cos\phi} \\ \hat{\alpha}_2(\vec{\xi}) &= \frac{4G}{c^2}\int_0^\infty d\xi'\xi'\Sigma(\xi')\int_0^{2\pi} d\phi \frac{-\xi'\sin\phi}{\xi^2 + \xi'^2 - 2\xi\xi'\cos\phi} & (2.12) \end{aligned}$$

By symmetry, the second component of the deflection angle is zero, therefore $\hat{\vec{\alpha}}$ is parallel to $\vec{\xi}$. Thus, using the lens equation, we find that also the vector $\vec{\eta}$ must be parallel to $\vec{\xi}$.

For the first component of the deflection angle in Eq. 2.12, the inner integral vanishes for $\xi' > \xi$, while it is $2\pi/\xi$ if $\xi' < \xi$. Then, the deflection angle for an axially symmetric lens is

$$\hat{\alpha}(\xi) = \frac{4G}{c^2}\frac{2\pi\int_0^\xi \Sigma(\xi')\xi' \, d\xi'}{\xi} = \frac{4GM(\xi)}{c^2\xi} \ . \tag{2.13}$$

The formula is similar to that derived for a point mass. The deflection is determined by the mass enclosed by the circle of radius ξ, $M(\xi)$.

2.3 Lensing potential

An extended distribution of matter is characterized by its *effective lensing potential*, obtained by projecting the three-dimensional Newtonian potential on the lens plane and by properly rescaling it:

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z) \mathrm{d}z \; . \tag{2.14}$$

The dimensionless counterpart of this function is given by

$$\Psi = \frac{D_{\mathsf{L}}^2}{\xi_0^2} \hat{\Psi} \; . \tag{2.15}$$

This lensing potential satisfies two important properties:

(1) the gradient of Ψ gives the scaled deflection angle:

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x}) \; . \tag{2.16}$$

Indeed,

$$\begin{aligned} \vec{\nabla}_x \Psi(\vec{x}) &= \xi_0 \vec{\nabla}_\perp \left(\frac{D_{\mathsf{LS}} D_{\mathsf{L}}}{\xi_0^2 D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(\vec{x}, z) \mathrm{d}z \right) && (2.17) \\ &= \frac{D_{\mathsf{LS}} D_{\mathsf{L}}}{\xi_0 D_{\mathsf{S}}} \frac{2}{c^2} \int \vec{\nabla}_\perp \Phi(\vec{x}, z) \mathrm{d}z && (2.18) \\ &= \vec{\alpha}(\vec{x}) && (2.19) \end{aligned}$$

(2) the Laplacian of Ψ gives twice the *convergence*:

$$\triangle_x \Psi(\vec{x}) = 2\kappa(\vec{x}) \; . \tag{2.20}$$

This is defined as a dimensionless surface density

$$\kappa(\vec{x}) \equiv \frac{\Sigma(\vec{x})}{\Sigma_{\mathsf{cr}}} \quad \text{with} \quad \Sigma_{\mathsf{cr}} = \frac{c^2}{4\pi G} \frac{D_{\mathsf{S}}}{D_{\mathsf{L}} D_{\mathsf{LS}}} \; , \tag{2.21}$$

where Σ_{cr} is called the *critical surface density*, a quantity which characterizes the lens system and which is a function of the angular diameter distances of lens and source.

Eq. 2.20 is derived from the Poisson equation,

$$\triangle \Phi = 4\pi G \rho \; . \tag{2.22}$$

The surface mass density is

$$\Sigma(\vec{\theta}) = \frac{1}{4\pi G} \int_{-\infty}^{+\infty} \triangle \Phi \mathrm{d}z \tag{2.23}$$

and

$$\kappa(\vec{\theta}) = \frac{1}{c^2} \frac{D_{\mathsf{L}} D_{\mathsf{LS}}}{D_{\mathsf{S}}} \int_{-\infty}^{+\infty} \triangle \Phi \mathrm{d}z \; . \tag{2.24}$$

Let us now introduce a two-dimensional Laplacian

$$\triangle_\theta = \frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2} = D_\mathsf{L}^2 \left(\frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} \right) = D_\mathsf{L}^2 \left(\triangle - \frac{\partial^2}{\partial z^2} \right) , \tag{2.25}$$

which gives

$$\triangle\Phi = \frac{1}{D_\mathsf{L}^2} \triangle_\theta \Phi + \frac{\partial^2 \Phi}{\partial z^2} . \tag{2.26}$$

Inserting Eq. 2.26 into Eq. 2.24, we obtain

$$\kappa(\vec{\theta}) = \frac{1}{c^2} \frac{D_\mathsf{LS}}{D_\mathsf{S} D_\mathsf{L}} \left[\triangle_\theta \int_{-\infty}^{+\infty} \Phi \mathrm{d}z + D_\mathsf{L}^2 \int_{-\infty}^{+\infty} \frac{\partial^2 \Phi}{\partial z^2} \mathrm{d}z \right] . \tag{2.27}$$

If the lens is gravitationally bound, $\partial\Phi/\partial z = 0$ at its boundaries and the second term on the right hand side vanishes. From Eqs. 2.14 and 2.15, we find

$$\kappa(\theta) = \frac{1}{2} \triangle_\theta \hat{\Psi} = \frac{1}{2} \frac{\xi_0^2}{D_\mathsf{L}^2} \triangle_\theta \Psi . \tag{2.28}$$

Since

$$\triangle_\theta = D_\mathsf{L}^2 \triangle_\xi = \frac{D_\mathsf{L}^2}{\xi_0^2} \triangle_x , \tag{2.29}$$

using adimensional quantities Eq. 2.28 reads

$$\kappa(\vec{x}) = \frac{1}{2} \triangle_x \Psi(\vec{x}) \tag{2.30}$$

Integrating Eq. (2.20), the effective lensing potential can be written in terms of the convergence as

$$\Psi(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \kappa(\vec{x}') \ln |\vec{x} - \vec{x}'| \mathrm{d}^2 x' , \tag{2.31}$$

from which we obtain that the scaled deflection angle is

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \mathrm{d}^2 x' \kappa(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|} . \tag{2.32}$$

2.4 Magnification and distortion

One of the main features of gravitational lensing is the distortion which it introduces into the shape of the sources. This is particularly evident when the source has no negligible apparent size. For example, background galaxies can appear as very long arcs in galaxy clusters.

The distortion arises because light bundles are deflected differentially. Ideally the shape of the images can be determined by solving the lens equation for all the points within the extended source. In particular, if the source is much smaller than the angular size on which the physical properties of the lens change, the relation between source and