2.3 Lensing potential

An extended distribution of matter is characterized by its *effective lensing potential*, obtained by projecting the three-dimensional Newtonian potential on the lens plane and by properly rescaling it:

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z)\mathrm{d}z \ . \tag{2.14}$$

This lensing potential satisfies two important properties:

(1) the gradient of $\hat{\Psi}$ gives the reduced deflection angle:

$$\vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) = \vec{\alpha}(\vec{\theta}) \ . \tag{2.15}$$

Indeed,

$$\begin{aligned}
\vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) &= D_{\mathsf{L}} \vec{\nabla}_\perp \hat{\Psi} = \vec{\nabla}_\perp \left(\frac{D_{\mathsf{LS}}}{D_{\mathsf{S}}} \frac{2}{c^2} \int \hat{\Phi}(\vec{\theta}, z)\mathrm{d}z \right) && (2.16)\\
&= \frac{D_{\mathsf{LS}}}{D_{\mathsf{S}}} \frac{2}{c^2} \int \vec{\nabla}_\perp \Phi(\vec{\theta}, z)\mathrm{d}z && (2.17)\\
&= \vec{\alpha}(\vec{\theta}) && (2.18)
\end{aligned}$$

Note that, using the adimensional notation,

$$\vec{\nabla}_x = \frac{\xi_0}{D_{\mathsf{L}}} \vec{\nabla}_\theta \ . \tag{2.19}$$

We can see that

$$\vec{\nabla}_x \hat{\Psi} = \frac{\xi_0}{D_{\mathsf{L}}} \vec{\nabla}_\theta \hat{\Psi} = \frac{\xi_0}{D_{\mathsf{L}}} \vec{\alpha} \ . \tag{2.20}$$

By multiplying both sides of this equation by D_{L}^2/ξ_0^2, we obtain

$$\frac{D_{\mathsf{L}}^2}{\xi_0^2} \vec{\nabla}_x \hat{\Psi} = \frac{D_{\mathsf{L}}}{\xi_0} \vec{\alpha} \ . \tag{2.21}$$

This allows us to introduce the dimensionless counterpart of $\hat{\Psi}$:

$$\Psi = \frac{D_{\mathsf{L}}^2}{\xi_0^2} \hat{\Psi} \ . \tag{2.22}$$

Substituting Eq. 2.22 into Eq 2.21, we see that

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x}) \ . \tag{2.23}$$

(2) the Laplacian of $\hat{\Psi}$ gives twice the *convergence*:

$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta}) \ . \tag{2.24}$$

This is defined as a dimensionless surface density

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\mathsf{cr}}} \quad \text{with} \quad \Sigma_{\mathsf{cr}} = \frac{c^2}{4\pi G} \frac{D_{\mathsf{S}}}{D_{\mathsf{L}} D_{\mathsf{LS}}} \ , \tag{2.25}$$

where Σ_{cr} is called the *critical surface density*, a quantity which characterizes the lens system and which is a function of the angular diameter distances of lens and source.

Eq. 2.24 is derived from the Poisson equation,

$$\triangle\Phi = 4\pi G\rho \ . \tag{2.26}$$

The surface mass density is

$$\Sigma(\vec{\theta}) = \frac{1}{4\pi G}\int_{-\infty}^{+\infty}\triangle\Phi \mathrm{d}z \tag{2.27}$$

and

$$\kappa(\vec{\theta}) = \frac{1}{c^2}\frac{D_{\mathsf{L}}D_{\mathsf{LS}}}{D_{\mathsf{S}}}\int_{-\infty}^{+\infty}\triangle\Phi \mathrm{d}z \ . \tag{2.28}$$

Let us now introduce a two-dimensional Laplacian

$$\triangle_\theta = \frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2} = D_{\mathsf{L}}^2\left(\frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2}\right) = D_{\mathsf{L}}^2\left(\triangle - \frac{\partial^2}{\partial z^2}\right) \ , \tag{2.29}$$

which gives

$$\triangle\Phi = \frac{1}{D_{\mathsf{L}}^2}\triangle_\theta\Phi + \frac{\partial^2\Phi}{\partial z^2} \ . \tag{2.30}$$

Inserting Eq. 2.30 into Eq. 2.28, we obtain

$$\kappa(\vec{\theta}) = \frac{1}{c^2}\frac{D_{\mathsf{LS}}}{D_{\mathsf{S}}D_{\mathsf{L}}}\left[\triangle_\theta\int_{-\infty}^{+\infty}\Phi \mathrm{d}z + D_{\mathsf{L}}^2\int_{-\infty}^{+\infty}\frac{\partial^2\Phi}{\partial z^2}\mathrm{d}z\right] \ . \tag{2.31}$$

If the lens is gravitationally bound, $\partial\Phi/\partial z = 0$ at its boundaries and the second term on the right hand side vanishes. From Eqs. 2.14 and 2.22, we find

$$\kappa(\theta) = \frac{1}{2}\triangle_\theta\hat{\Psi} = \frac{1}{2}\frac{\xi_0^2}{D_{\mathsf{L}}^2}\triangle_\theta\Psi \ . \tag{2.32}$$

Since

$$\triangle_\theta = D_{\mathsf{L}}^2\triangle_\xi = \frac{D_{\mathsf{L}}^2}{\xi_0^2}\triangle_x \ , \tag{2.33}$$

using adimensional quantities Eq. 2.32 reads

$$\kappa(\vec{x}) = \frac{1}{2}\triangle_x\Psi(\vec{x}) \tag{2.34}$$

Integrating Eq. (2.24), the effective lensing potential can be written in terms of the convergence as

$$\Psi(\vec{x}) = \frac{1}{\pi}\int_{\mathsf{R}^2}\kappa(\vec{x}')\ln|\vec{x}-\vec{x}'|\mathrm{d}^2x' \ , \tag{2.35}$$

from which we obtain that the scaled deflection angle is

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi}\int_{\mathsf{R}^2}\mathrm{d}^2x'\kappa(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} \ . \tag{2.36}$$

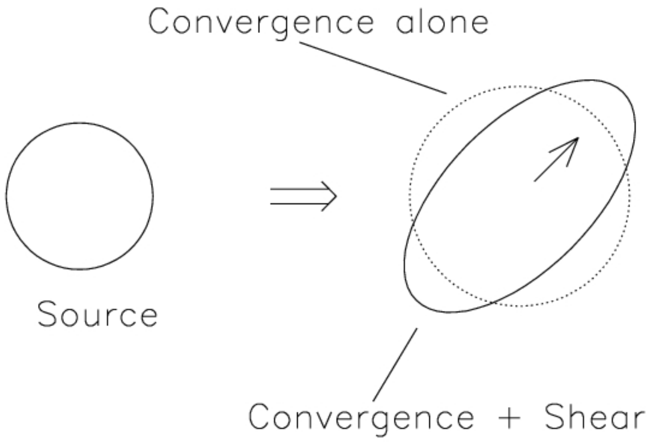

Figure 2.2: Distortion effects due to convergence and shear on a circular source (Figure from Narayan & Bartelmann, 1995).

2.4 Magnification and distortion

One of the main features of gravitational lensing is the distortion which it introduces into the shape of the sources. This is particularly evident when the source has no negligible apparent size. For example, background galaxies can appear as very long arcs in galaxy clusters.

The distortion arises because light bundles are deflected differentially. Ideally the shape of the images can be determined by solving the lens equation for all the points within the extended source. In particular, if the source is much smaller than the angular size on which the physical properties of the lens change, the relation between source and image positions can locally be linearized. In other words, the distortion of images can be described by the Jacobian matrix

$$A \equiv \frac{\partial \vec{y}}{\partial \vec{x}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{x})}{\partial x_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{x})}{\partial x_i \partial x_j} \right) \ , \tag{2.37}$$

where x_i indicates the i-component of $\vec{x}$ on the lens plane. Eq. (2.37) shows that the elements of the Jacobian matrix can be written as combinations of the second derivatives of the lensing potential.

For brevity, we will use the shorthand notation

$$\frac{\partial^2 \Psi(\vec{x})}{\partial x_i \partial x_j} \equiv \Psi_{ij} \ . \tag{2.38}$$

We can now split off an isotropic part from the Jacobian:

$$\left(A - \frac{1}{2}\mathrm{tr}A \cdot I\right)_{ij} = \delta_{ij} - \Psi_{ij} - \frac{1}{2}(1 - \Psi_{11} + 1 - \Psi_{22})\delta_{ij} \tag{2.39}$$

$$= -\Psi_{ij} + \frac{1}{2}(\Psi_{11} + \Psi_{22})\delta_{ij} \tag{2.40}$$

$$= \begin{pmatrix} -\frac{1}{2}(\Psi_{11} - \Psi_{22}) & -\Psi_{12} \\ -\Psi_{12} & \frac{1}{2}(\Psi_{11} - \Psi_{22}) \end{pmatrix} . \tag{2.41}$$

This is manifestly an antisymmetric, trace-free matrix is called the shear matrix. It quantifies the projection of the gravitational tidal field (the gradient of the gravitational force), which describes distortions of background sources.

This allows us to define the pseudo-vector $\vec{\gamma} = (\gamma_1, \gamma_2)$ on the lens plane, whose components are

$$\gamma_1(\vec{x}) = \frac{1}{2}(\Psi_{11} - \Psi_{22}) \tag{2.42}$$

$$\gamma_2(\vec{x}) = \Psi_{12} = \Psi_{21} , \tag{2.43}$$

This is called the *shear*.

The eigenvalues of the shear matrix are

$$\pm\sqrt{\gamma_1^2 + \gamma_2^2} = \pm\gamma . \tag{2.44}$$

Thus, there exists a coordinate rotation by an angle ϕ such that

$$\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} = \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \tag{2.45}$$

Remark:

Note the factor 2 on the angle ϕ, which reminds that the shear component are elements of a 2×2 tensor and not a vector.

The remainder of the Jacobian is

$$\frac{1}{2}\mathrm{tr}A \cdot I = \left[1 - \frac{1}{2}(\Psi_{11} + \Psi_{22})\right]\delta_{ij} \tag{2.46}$$

$$= \left(1 - \frac{1}{2}\triangle\Psi\right)\delta_{ij} = (1 - \kappa)\delta_{ij} . \tag{2.47}$$

Thus, the Jacobian matrix becomes

$$A = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}$$

$$= (1 - \kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} . \tag{2.48}$$

The last equation explains the meaning of both convergence and shear. The transformation induced by the convergence is isotropic, i.e. the images are only rescaled by a constant factor in all directions. On the other hand, the shear stretches the intrinsic shape of the source along one privileged direction.The Jacobian matrix has two eigenvalues,

$$\lambda_t = 1 - \kappa - \gamma \tag{2.49}$$

$$\lambda_r = 1 - \kappa + \gamma . \tag{2.50}$$

Let consider the reference frame where the Jacobian is diagonal. Then,

$$A = \begin{pmatrix} 1-\kappa-\gamma & 0 \\ 0 & 1-\kappa+\gamma \end{pmatrix} . \tag{2.51}$$

Consider a circular source, whose isophotes have equation $y_1^2 + y_2^2 = r^2$. The lens equation implies that the points on the source plane satisfying this equation are mapped onto the points (x_1, x_2), such that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma & 0 \\ 0 & 1-\kappa+\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} . \tag{2.52}$$

Thus

$$y_1 = (1-\kappa-\gamma)x_1 \tag{2.53}$$

$$y_2 = (1-\kappa+\gamma)x_2 . \tag{2.54}$$

Summing in quadrature, we obtain

$$r^2 = y_1^2 + y_2^2 = (1-\kappa-\gamma)^2 x_1^2 + (1-\kappa+\gamma)^2 x_2^2 , \tag{2.55}$$

which is the equation of an ellipse on the lens plane. Thus, a circular source, which is small enough compared to the scale of the lens, like that shown in Fig. (2.2) is mapped into an ellipse when κ and γ are both non-zero. The semi-major and -minor axes are

$$a = \frac{r}{1-\kappa-\gamma} \quad , \quad b = \frac{r}{1-\kappa+\gamma} . \tag{2.56}$$

Obviously, the ellipse reduces to a circle if $\gamma = 0$.

An important consequence of the lensing distortion is the magnification. Through the lens equation, the solid angle element $\delta\beta^2$ (or equivalently the surface element δy^2) is mapped into the solid angle $\delta\theta^2$ (or in the surface element δx^2). Since the Liouville theorem and the absence of emission and absorbtion of photons in gravitational light deflection ensure the conservation of the source surface brightness, the change of the solid angle under which the source is seen implies that the flux received from a source is magnified (or demagnified).

Given Eq. (2.37), the *magnification* is quantified by the inverse of the determinant of the Jacobian matrix. For this reason, the matrix $M = A^{-1}$ is called the *magnification tensor*. We therefore define

$$\mu \equiv \det M = \frac{1}{\det A} = \frac{1}{(1-\kappa)^2 - \gamma^2} . \tag{2.57}$$

The eigenvalues of the magnification tensor (or the inverse of the eigenvalues of the Jacobian matrix) measure the amplification in the tangential and in the radial direction and are given by

$$\mu_{\mathrm{t}} = \frac{1}{\lambda_{\mathrm{t}}} = \frac{1}{1-\kappa-\gamma} \tag{2.58}$$

$$\mu_{\mathrm{r}} = \frac{1}{\lambda_{\mathrm{r}}} = \frac{1}{1-\kappa+\gamma} . \tag{2.59}$$

The magnification is ideally infinite where $\lambda_{\mathrm{t}} = 0$ and where $\lambda_{\mathrm{r}} = 0$. These two conditions define two curves in the lens plane, called the *tangential* and the *radial critical line*, respectively. An image forming along the tangential critical line is strongly distorted tangentially to this line. On the other hand, an image forming close to the radial critical line is stretched in the direction perpendicular to the line itself.