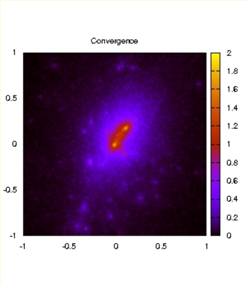
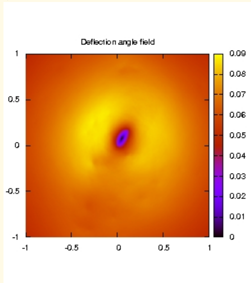
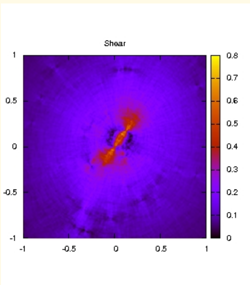
Example: Numerically simulated galaxy cluster



Galaxy clusters are the most massive bound objects in the Universe. They are "young" structures, whose assembling process is still on-going. For this reason they are characterized by an high level of complexity. The luminous matter within them (gas and stars) accounts for $\sim 10\%$ of their mass. The rest is dark matter. Numerical simulations provide the most realistic description of these cosmic structures. N-body and hydrodynamical simulations have been used to simulate the formation and the evolution of systems on different scales. The figure on the left side shows the adimensional surface mass density (or *convergence*) of a cluster-sized dark matter halo simulated at $z \sim 0.3$. The mass of such object is $\sim 10^{15} M_{\odot}$.

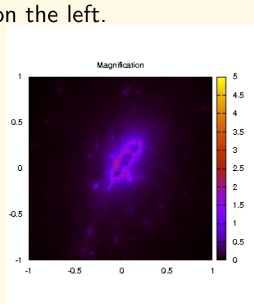
Imagine that a bundle of light rays passes through the mass distribution showed above. Each mass element of the lens contributes to deflect the light coming from background sources. Eq.2.3 allows to calculate the deflection angle at each position $\vec{\xi}$ on the lens plane. The resulting deflection angle field is shown on the right.

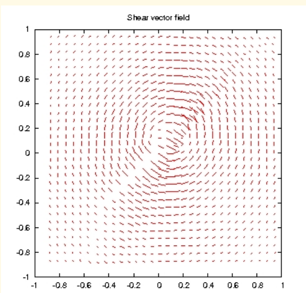


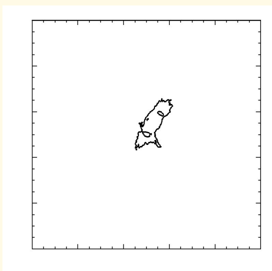


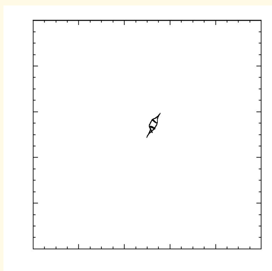
The lensing effect can be decomposed into two terms: the isotropic term given by the convergence and the anisotropic term given by the shear γ. This is a pseudo-vector, whose orientation define the direction into which an image is stretched. All around the cluster, the shear tends to be tangential to the lens iso-density contours (see the left panel below). Close to the cluster cores, images can be distorted also towards the cluster center. The intensity of γ determines the amplitude of the distortion. The shear pattern for our numerical cluster is shown on the left.

By distorting them, the lens magnifies the sources. Depending on where the sources are located behind the cluster, the resulting magnification is different. In the Figure on the right shown is the magnification on the lens plane (in logarithmic scale!). It is ideally infinite along the so-called *lens critical lines*. The sources generating images around the critical lines are located along the *caustics*. The critical lines and the caustics are shown in the middle and in the right panels below, respectively.









2.5 Lensing to the second order

In section 2.4, we discussed the effects of lensing at the first order. We briefly mention now some second order effects. Using a Taylor expansion around the origin, the second order lens equation can be written as

$$\beta_i \simeq \frac{\partial \beta_i}{\partial \theta_j}\theta_j + \frac{1}{2}\frac{\partial^2 \beta_i}{\partial \theta_j \partial \theta_k}\theta_j\theta_k \ . \tag{2.60}$$

The lensing effect is described at the first order by the Jacobian matrix A. Now, we introduce the tensor

$$D_{ijk} = \frac{\partial^2 \theta_i}{\partial \theta_j \partial \theta_k} = \frac{\partial A_{ij}}{\partial \theta_k} \ . \tag{2.61}$$

Then, Eq. 2.60 reads

$$\theta_i \simeq A_{ij}\theta_j + \frac{1}{2}D_{ijk}\theta_j\theta_k \tag{2.62}$$

By simple algebra, it can be shown that

$$D_{ij1} = \begin{pmatrix} -2\gamma_{1,1} - \gamma_{2,2} & -\gamma_{2,1} \\ -\gamma_{2,1} & -\gamma_{2,2} \end{pmatrix} , \tag{2.63}$$

and

$$D_{ij2} = \begin{pmatrix} -\gamma_{2,1} & -\gamma_{2,2} \\ -\gamma_{2,2} & 2\gamma_{1,2} - \gamma_{2,1} \end{pmatrix} . \tag{2.64}$$

Thus, the second order lensing effect can be expressed in terms of the derivatives of the shear (or in terms of the third derivatives of the potential).

2.5.1 Complex notation

It is quite useful to use complex notation to map vectors or pseudo-vectors on the complex plane. Indeed, in this case we can also use complex differential operators to write down some relations between the lensing quantities in a very concise way.

In complex notation, any vector $v = (v_1, v_2)$ is written as

$$v = v_1 + iv_2 \ ; . \tag{2.65}$$

Similarly we can define the complex deflection angle $\alpha = \alpha_1 + i\alpha_2$ and the complex shear $\gamma = \gamma_1 + i\gamma_2$.

It is also possible to define some complex differential operators, namely

$$\partial = \partial_1 + i\partial_2 \tag{2.66}$$

and

$$\partial^\dagger = \partial_1 - i\partial_2 \ . \tag{2.67}$$

Using this formalism, we can easily see that

$$\partial\hat{\Psi} = \partial_1\hat{\Psi} + i\partial_2\hat{\Psi} = \alpha_1 + i\alpha_2 = \alpha \ . \tag{2.68}$$