6 Lensing by large-scale structures

6.1 Light propagation through an inhomogeneous universe

In unperturbed spacetime, light travels alog null geodesic lines of the symmetric, homogenous and isotropic Friedmann-Lemaitre space-time.

In contrast to the earlier treatment, we have to take into account that lenses can now be of comparable size to the curvature scale of the universe, thus we need to refine the picture of straight light paths which are instantly deflected by sheet-like, thin lenses.

Starting from null geodesic in space-time, it can be shown that light rays propagate through the unperturbed Friedmann-Lemaitre spacetime such that the comoving separation vector $\vec{x}$ between them changes with the radial coordinate w as

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}w^2} + K\vec{x} = 0 \; , \tag{6.1}$$

where $K = (H_0/c)^2(\Omega_0 + \Omega_\Lambda - 1)$ is the curvature parameter of the universe. Note that c/H_0 is the Hubble length, so K has the unit of an inverse squared length, as it has to be.

Ω_0 is the density parameter of the universe today,

$$\Omega_0 = \left(\frac{3H_0^2}{8\pi G}\right)^{-1} \rho_0 \; , \tag{6.2}$$

while Ω_Λ is the density parameter corresponding to the cosmological constant,

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} \; . \tag{6.3}$$

H_0 is the Hubble constant. According to present knowledge, $\Omega_0 \approx 0.3$, $\Omega_\Lambda \approx 0.7$ and $K \approx 0$.

Comoving means that the physical separation $\vec{r}$ between the rays is divided by the scale factor of the universe,

$$\vec{x} = \frac{\vec{r}}{a} \tag{6.4}$$

in order to get rid of the expansion of space-time.

The metric is written as

$$ds^2 = c^2dt^2 - a^2[dw^2 + f_K^2(w)d^2\Omega] \; , \tag{6.5}$$

such that dw is the radial, comoving distance element, and $f_K(w)$ is given by

$$f_K(w) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}w) & (K > 0) \\ w & (K = 0) \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}w) & (K < 0) \end{cases} \tag{6.6}$$

The propagation equation is easily solved. It is an oscillator equation, so that its general solution is

$$\vec{x} = \vec{A}\cos\sqrt{K}w + \vec{B}\sin\sqrt{K}w \qquad (K > 0) . \tag{6.7}$$

With the boundary conditions $\vec{x}(w=0) = 0$ and $\mathrm{d}\vec{x}/\mathrm{d}w|_{w=0} = \vec{\theta}$, we find

$$\vec{x}(w) = \vec{\theta}\frac{1}{\sqrt{K}}\sin\sqrt{K}w . \tag{6.8}$$

Generally, for negative and vanishing K, we find

$$\vec{x}(w) = \vec{\theta} f_K(w) . \tag{6.9}$$

These solutions have a very simple interpretation: obviously, for $K = 0$, $\vec{x} = \vec{\theta}w$, as we know in Euclidean space. For positive or negative curvature, the light rays approach each other, or depart from each other compared to the flat case, as the meridional lines on a sphere or a hyperboloid do.

Adding perturbations is simple considering that the lensing masses are typically much smaller than the Hubble radius. Then, space-time can be considered flat in their surroundings, and we can use our earlier result on the deflection angle in the form

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}w^2} = -\frac{2}{c^2}\vec{\nabla}_\perp\phi , \tag{6.10}$$

where it must now be noted that the perpendicular gradient of ϕ must be taken with respect to the comoving coordinates as well. This means

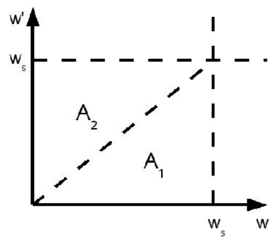
$$\vec{\nabla}_\perp\phi = \frac{1}{f_K(w)}\vec{\nabla}_{\vec{\theta}}\phi . \tag{6.11}$$

The propagation equation changes to

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}w^2} + K\vec{x} = -\frac{2}{c^2}\vec{\nabla}_\perp\phi , \tag{6.12}$$

which now incorporates overall space-time curvature and local perturbations caused by a potential ϕ.

The inhomogeneus oscillator equation can be solved by constructing a Greens function $G(w, w')$, which is defined on the square $0 \leq w \leq w_s$, $0 \leq w' \leq w_s$, where w_s is the coordinate distance to the source.



According to the definition of a Green function, $G(w, w')$ must satisfy the following conditions

- $G(w, w')$ is continously differentaible in both triangles $A_{1,2}$ and satisfies the homogeneous differential equation

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}w^2} + K\vec{x} = 0 \; ; \tag{6.13}$$

- $G(w, w')$ is continous on the entire square;
- the derivative of $G(w, w')$ with respect to w jumps by 1 on the boundary between A_1 and A_2;
- as a function of w, $G(w, w')$ satisfies the homogeneous boundary conditions on the solution.

Accordingly, we set up

$$G(w, w') = \begin{cases} A(w')\cos\sqrt{K}w + B(w')\sin\sqrt{K}w & \text{on } A_1 \\ C(w')\cos\sqrt{K}w + D(w')\sin\sqrt{K}w & \text{on } A_2 \end{cases} \; . \tag{6.14}$$

The homogeneous boundary conditions demand $A = B = 0$.
Continuity requires

$$C\cos\sqrt{K}w' + D\sin\sqrt{K}w' = 0 \; , \tag{6.15}$$

and the jump in the derivative implies

$$-C\sin\sqrt{K}w' + D\cos\sqrt{K}w' = \frac{1}{\sqrt{K}} \; . \tag{6.16}$$

Thus,

$$C = -\frac{1}{\sqrt{K}}\sin\sqrt{K}w' \tag{6.17}$$

$$D = \frac{1}{\sqrt{K}}\cos\sqrt{K}w' \; . \tag{6.18}$$

This implies

$$G(w, w') = \begin{cases} 0 & (w < w') \\ \frac{1}{\sqrt{K}}\sin\sqrt{K}(w - w') & (w > w') \end{cases} \; . \tag{6.19}$$

More generally, i.e. for arbitrary sign of K, we find instead

$$G(w, w') = \begin{cases} 0 & (w < w') \\ f_K(w - w') & (w > w') \end{cases} \; . \tag{6.20}$$

Therefore the general solution of the propagation equation reads

$$\vec{x} = f_K(w)\vec{\theta} - \frac{2}{c^2}\int_0^w \mathrm{d}w' f_K(w - w')\vec{\nabla}_\perp\phi \; . \tag{6.21}$$

As in the single-lens plane approach, we evaluate this integral along the unperturbed path $f_K(w)\vec{\theta}$.
The deflection angle is defined as the deviation between the perturbed and the unperturbed path,

$$\vec{\alpha} = \frac{f_K(w)\vec{\theta} - \vec{x}}{f_K(w)} = \frac{2}{c^2}\int_0^w \mathrm{d}w' \frac{f_K(w - w')}{f_K(w)}\vec{\nabla}_\perp\phi[f_K(w')\vec{\theta}, w'] \; . \tag{6.22}$$

This is now the deflection angle accumulated along a light path propagating into direction $\vec{\theta}$ out to the coordinate distance w. Hence, we denote it as $\vec{\alpha}(\vec{\theta}, w)$.

For a spatially flat universe, $K = 0$ and $f_K(w) = w$. Then,

$$\begin{aligned} \vec{\alpha}(\vec{\theta}, w) &= \frac{2}{c^2}\int_0^w \mathrm{d}w' \left(1 - \frac{w'}{w}\right)\vec{\nabla}_\perp \phi(w'\vec{\theta}, w') \\ &= \frac{2w}{c^2}\int_0^1 \mathrm{d}y(1-y)\vec{\nabla}_\perp\phi(wy\vec{\theta}, wy) \;. \end{aligned} \tag{6.23}$$

6.2 Effective convergence

In the single lens-plane case, the convergence is one half the divergence of $\vec{\alpha}$. Analogously, we define here an effective convergence for large-scale structure lenses,

$$\begin{aligned} \kappa_{\text{eff}}(\vec{\theta}, w) &= \frac{1}{2}\vec{\nabla}_{\vec{\theta}}\vec{\alpha}(\vec{\theta}, w) \\ &= \frac{1}{c^2}\int \mathrm{d}w' \frac{f_K(w')f_K(w-w')}{f_K(w)}\triangle^{(2)}\phi[f_K(w')\vec{\theta}', w'] \;, \end{aligned} \tag{6.24}$$

where $\triangle^{(2)}$ is the two-dimensional Laplacian with respect to comoving coordinates,

$$\triangle^{(2)} = \vec{\nabla}_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \;. \tag{6.25}$$

We now do the same as we did when we introduced the lensing potential: we replace

$$\triangle^{(2)} \to \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \;, \tag{6.26}$$

and assume that $\partial\phi/\partial z = 0$ at the boundaries of the perturbations. Then, we can write

$$\kappa_{\text{eff}} = \frac{1}{c^2}\int_0^w \mathrm{d}w' \frac{f_K(w')f_K(w-w')}{f_K(w)}\triangle\phi(f_K(w')\vec{\theta}, w') \;, \tag{6.27}$$

and substitute for $\triangle\phi$ using Poisson's equation.

In its original form, Poisson's equation reads

$$\triangle_r\phi = 4\pi G\rho \;, \tag{6.28}$$

where the Laplacian is now taken with respect to physical coordinates. Introducing the density contrast

$$\delta \equiv \frac{\rho - \overline{\rho}}{\overline{\rho}} \;, \tag{6.29}$$

we can write

$$\triangle\phi = 4\pi G\overline{\rho}(1+\delta)a^2 = 4\pi G\overline{\rho}_0 a^{-1}(1+\delta) \;, \tag{6.30}$$

where we have inserted $\overline{\rho}a^{-3}$ as for ordinary (non-relativistic) matter. Decoupling the potential into a background potential

$$\triangle\overline{\phi} = 4\pi G\overline{\rho}_0 a^{-1} \;, \tag{6.31}$$

and a peculiar (perturbing) potential ϕ, we have

$$\triangle\phi = 4\pi G\overline{\rho}_0 a^{-1}\delta \;. \tag{6.32}$$

Using further

$$\overline{\rho}_0 = \Omega_0 \frac{3H_0^2}{8\pi G} \tag{6.33}$$

yields the Poisson equation that we need,

$$\triangle\phi = \frac{3}{2} H_0^2 \Omega_0 \frac{\delta}{a} \;. \tag{6.34}$$

The effective convergence can then be written as

$$\kappa_{\rm eff}(\vec\theta, w) = \frac{3\Omega_0}{2}\left(\frac{H_0}{c}\right)^2 \int_0^w {\rm d}w' \frac{f_K(w')f_K(w-w')}{f_K(w)} \frac{\delta[f_K(w')\vec\theta, w']}{a} \;. \tag{6.35}$$

Notice the similarity of the distance factor with the factor $D_{\sf L}D_{\sf LS}/D_{\sf S}$ that we had in the single-lens case.

If the sources are distributed in redshift or, equivalently, in coordinate distance w, the mean effective convergence is

$$\langle\kappa_{\rm eff}\rangle(\vec\theta) = w\int_0^{w_H} {\rm d}w G(w)\kappa_{\rm eff}(\vec\theta, w) \;, \tag{6.36}$$

where $G(w){\rm d}w$ is the probability to find a source within ${\rm d}w$ of w. Then we can write

$$\langle\kappa_{\rm eff}\rangle(\vec\theta) = \frac{3H_0^2\Omega_0}{2c^2}\int_0^{w_H} {\rm d}w W(w) f_K(w) \frac{\delta[f_K(w)\vec\theta, w]}{a(w)} \;, \tag{6.37}$$

with the effective weight function

$$W(w) = \int_w^{w_H} {\rm d}w' G(w') \frac{f_K(w'-w)}{f_K(w')} \;. \tag{6.38}$$

6.3 Limber's equation

It is impossible to predict exactly which density fluctuations a light ray will find on its way. Concerning the effective convergence, we thus need a statistical approach.

We want to compute the correlation function

$$\langle\kappa(\vec\theta)\kappa(\vec\theta+\vec\phi)\rangle_{\vec\theta} = \xi_\kappa(\phi) \;, \tag{6.39}$$

in which the average extends over all positions $\vec\theta$ on the sky, and over all directions of the separation vector $\vec\phi$. Due to isotropy, the result cannot depend on the direction of $\vec\phi$.

It is typically more convenient to go into Fourier space and to use the power spectrum instead. Suppose we have a function $g(x)$ of n-dimensional space, whose correlation function is

$$\xi_{gg}(y) = \langle g(x)g(x+y)\rangle_x \;. \tag{6.40}$$

We Fourier transform $g(x)$,

$$\hat{g}(k) = \int \mathrm{d}^n x g(x) \exp(ikx) \qquad g(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \hat{g}(k) \exp(-ikx) \tag{6.41}$$

and compute the correlation function in Fourier space,

$$\begin{aligned} \langle \hat{g}(k)\hat{g}^*(k')\rangle &= \left\langle \int \mathrm{d}^n x g(x) \exp(ikx) \int \mathrm{d}^n x' g(x') \exp(-ik'x') \right\rangle \\ &= \int \mathrm{d}^n x \exp(ikx) \int \mathrm{d}^n x' \exp(-ik'x') \langle g(x) g(x')\rangle \; . \end{aligned} \tag{6.42}$$

Inserting $y + x = x'$ and using the isotropy of the correlation function, we can continue to compute

$$\begin{aligned} \langle \hat{g}(k)\hat{g}^*(k')\rangle &= \int \mathrm{d}^n x \exp[i(k-k')x] \int \mathrm{d}^n y \exp(-ik'y) \xi_{gg}(y) \\ &= (2\pi)^n \delta_D^{(n)}(k-k') P_g(k) \; , \end{aligned} \tag{6.43}$$

where we have defined the power spectrum

$$P_g(k) \equiv \int \mathrm{d}^n y \exp(-iky) \xi_{gg}(y) \tag{6.44}$$

as the Fourier transform of the correlation function. $\delta_D^{(n)}$ is the Dirac delta function in n dimensions.

Suppose we are given the power spectrum of a three-dimensional function $\delta(\vec{x})$. What is the power spectrum of a two-dimensional projection

$$g(\vec{\theta}) = \int \mathrm{d}w q(w) \delta[f_K(w)\theta, w] \; , \tag{6.45}$$

where $q(w)$ is a weighting function?

Its correlation function is

$$\begin{aligned} \xi_{gg} &= \langle g(\vec{\theta}) g(\vec{\theta}')\rangle \\ &= \int q(w)\mathrm{d}w \int q(w')\mathrm{d}w' \langle \delta[f_K(w)\vec{\theta}, w] \delta[f_K(w')\vec{\theta}, w']\rangle \; . \end{aligned} \tag{6.46}$$

Inserting the Fourier transform of δ, we find

$$\begin{aligned} \xi_{gg} &= \int q(w)\mathrm{d}w \int q(w')\mathrm{d}w' \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \langle \hat{\delta}(\vec{k}) \hat{\delta}^*(\vec{\kappa}')\rangle \exp(-if_K(w)\vec{\theta}\vec{k}_\perp) \\ & \quad \exp(if_K(w')\vec{\theta}'\vec{k}'_\perp) \exp(-ik_s w) \exp(ik'_s w') \; , \end{aligned} \tag{6.47}$$

where we have split the wave vector $\vec{k}$ into a perpendicular and a parallel part, $\vec{k}_\perp$ and k_s, respectively. The average over $\hat{\delta}\hat{\delta}^*$ can be replaced by the power spectrum of δ,

$$\begin{aligned} \xi_{gg} &= \int q(w)\mathrm{d}w \int q(w')\mathrm{d}w' \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P_\delta(k) \exp[-i(f_K(w)\vec{\theta} - f_K(w')\vec{\theta}')\vec{k}_\perp] \\ & \quad \cdot \exp[-ik_s(w-w')] \\ &= \int q^2(w)\mathrm{d}w \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P_\delta(k) \exp[-if_k(w)(\vec{\theta} - \vec{\theta}'\vec{k}_\perp] \\ & \quad \cdot \int \mathrm{d}(\Delta w) \exp(-ik_s \Delta w) \; . \end{aligned} \tag{6.48}$$

The last factor,

$$\int \mathrm{d}(\Delta w) \exp\left(-ik_s \Delta w\right) = 2\pi\delta_D(k_s)\ , \tag{6.49}$$

means that only such modes contribute which are perpendicular to the line-of-sight, $\vec{k} = (\vec{k}_\perp, 0)$.

The k_s-integral can the be carried out and we get

$$\xi_{gg} = \int q^2(w)\mathrm{d}w \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} P_\delta(|\vec{k}_\perp|) \exp\left(i f_K(w)(\vec{\theta} - \vec{\theta}')\vec{k}_\perp\right)\ . \tag{6.50}$$

The difference $\vec{\theta} - \vec{\theta}'$ is the separation vector between the two rays. Defining $\phi \equiv |\vec{\theta} - \vec{\theta}'|$ and using isotropy, we get

$$\begin{aligned} \xi_{gg}(\phi) &= \int q^2(w)\mathrm{d}w \int \frac{\mathrm{d}^2 k}{(2\pi)^2} P_\delta(k) \exp\left(-i f_K(w)\vec{k}\vec{\phi}\right) \\ &= \int q^2(w)\mathrm{d}w \int \frac{k\mathrm{d}k}{(2\pi)} P_\delta(k) J_0[f_K(w)\phi k]\ . \end{aligned} \tag{6.51}$$

The power spectrum of the projected quantity $g(\vec{\theta})$ is

$$\begin{aligned} P_g(l) &= \int \mathrm{d}^2\phi\, \xi_{gg}(\phi) \exp\left(i\vec{l}\vec{\phi}\right) \\ &= \int q^2(w)\mathrm{d}w \int \frac{\mathrm{d}^2 k}{(2\pi)^2} P_\delta(k) \exp\left\{i[\vec{l} - f_K(w)\vec{k}]\vec{\phi}\right\} \\ &= \int \mathrm{d}w \frac{q^2(w)}{f_K^2(w)} P_\delta\left(\frac{l}{f_K(w)}\right)\ . \end{aligned} \tag{6.52}$$

We can now simply read the power spectrum of the effective convergence off the expression for κ_{eff}. With

$$q(w) = \frac{3H_0^2\Omega_0}{2c^2} W(w) f_K(w) \frac{1}{a}\ , \tag{6.53}$$

we have

$$P_\kappa(l) = \frac{9H_0^4\Omega_0^2}{4c^2} \int_0^{w_H} \frac{W^2(w)}{a^2} P_\delta\left(\frac{l}{f_K(w)}\right) \mathrm{d}w\ . \tag{6.54}$$

This power spectrum will be central to all further considerations.

For example, the convergence correlation function is

$$\begin{aligned} \xi_\kappa(\phi) &= \int \frac{\mathrm{d}^2 l}{(2\pi)^2} P_\kappa(l) \exp\left(-i\vec{l}\vec{\phi}\right) \\ &= \int \frac{l\mathrm{d}l}{2\pi} P_\kappa(l) J_0(l\phi)\ . \end{aligned} \tag{6.55}$$

The magnification is, as in the single lens-plane case,

$$\mu = \frac{1}{\det A} \qquad A = I - \frac{\partial\vec{\alpha}}{\partial\vec{\theta}} \tag{6.56}$$

$$\Rightarrow \qquad \mu \approx 1 + \vec{\nabla}_{\vec{\theta}} \cdot \vec{\alpha} = 1 + 2\kappa_{\mathrm{eff}}\ . \tag{6.57}$$

Thus, the magnification fluctuation, i.e. its deviation from unity, has the correlation function

$$\langle\delta\mu(\vec{\theta})\delta\mu(\vec{\theta}+\vec{\phi})\rangle = \xi_\mu(\phi) = 4\xi_\kappa(\phi) \; . \tag{6.58}$$

Its r.m.s. value is

$$\langle\delta\mu^2\rangle^{1/2} = \xi_\mu^{1/2}(0) = \left[\int_0^\infty \frac{l\mathrm{d}l}{2\pi} P_\kappa(l)\right]^{1/2} \; , \tag{6.59}$$

which gives the typical magnification of cosmic sources by large-scale structures.

6.4 Shear correlation functions

Compared to the convergence, the shear depends on the direction with respect to which it is defined. Let ψ be the effective lensing potential and the separation vector $\vec{\phi}$ between any two points have a polar angle α. Then, the tangential component of the shear with respect to that direction is

$$\gamma_t = \gamma(\cos^2\alpha - \sin^2\alpha) = \gamma\cos 2\alpha \; , \tag{6.60}$$

if α is the polar angle with respect to the principal-axis frame of the shear. Of course, α will vary, so we have to average over it.

Let us now define the correlation function of the tangential shear,

$$\langle\gamma_t\gamma_t'\rangle \equiv \xi_{tt}(\phi) \; , \tag{6.61}$$

which can be obtained from the power spectrum of the tangential shear component,

$$\xi_{tt}(\phi) = \int \frac{\mathrm{d}^2 l}{(2\pi)^2} P_{\gamma_t}(l) \exp\left(-i\vec{l}\vec{\phi}\right) \; . \tag{6.62}$$

According to its definition, the tangential component of the shear has the Fourier transform

$$\begin{aligned} \hat{\gamma}_t &= -\frac{1}{2}(k_1^2 + k_2^2)\hat{\psi} \\ &= \frac{k^2}{2}(\cos^2\alpha - \sin^2\alpha)\hat{\psi} \; . \end{aligned} \tag{6.63}$$

Thus, its power spectrum is

$$P_{\gamma_t} = \frac{k^4}{4}(\cos^2\alpha - \sin^2\alpha)^2 P_\psi \; . \tag{6.64}$$

We know the power spectrum of κ,

$$P_\kappa = \frac{1}{4}(k_1^2 + k_2^2)^2 P_\psi = \frac{k^4}{4} P_\psi \; , \tag{6.65}$$

so that we can infer P_{γ_t}:

$$P_{\gamma_t} = (\cos^2\alpha - \sin^2\alpha)^2 P_\kappa \; . \tag{6.66}$$

Now, $(\cos^2\alpha - \sin^2\alpha) = \cos^2 2\alpha = 1/2(1 + \cos 4\alpha)$, from which we find

$$\langle\gamma_t\gamma_t'\rangle = \frac{1}{2}\int \frac{l\mathrm{d}l}{2\pi} P_\kappa(l)[J_0(l\phi) + J_4(l\phi)] \; . \tag{6.67}$$

Similarly, the "cross-component" of the shear is

$$\gamma_x = \gamma \sin 2\alpha \;, \tag{6.68}$$

and its autocorrelation function is

$$\langle \gamma_x \gamma_x' \rangle = \xi_{xx}(\phi) = \int \frac{\mathrm{d}^2 l}{(2\pi)^2} P_{\gamma_x}(l) \exp\left(-i\vec{l}\vec{\phi}\right) \;. \tag{6.69}$$

As before, we find

$$\begin{aligned} P_{\gamma_x} &= k_1^2 k_2^2 P_\psi = k^4 \cos^2\alpha \sin^2\alpha P_\psi \\ &= 4\cos^2\alpha \sin^2\alpha P_\kappa \;. \end{aligned} \tag{6.70}$$

Since

$$\begin{aligned} 4\cos^2\alpha \sin^2\alpha &= 1 - \cos^2 2\alpha = 1 - \frac{1}{2} - \frac{1}{2}\cos 4\alpha \\ &= \frac{1}{2}(1 - \cos 4\alpha) \;, \end{aligned} \tag{6.71}$$

we find

$$\langle \gamma_x \gamma_x' \rangle = \frac{1}{2} \int \frac{l\mathrm{d}l}{2\pi} P_\kappa(l) [J_0(l\phi) - J_4(l\phi)] \;. \tag{6.72}$$

Finally, the mixed correlation function,

$$\langle \gamma_t \gamma_x' \rangle \tag{6.73}$$

follows from the mixed power spectrum,

$$\begin{aligned} P_{\gamma_t \gamma_x} &= \frac{1}{2}(k_1^2 - k_2^2) k_1 k_2 \frac{4}{k^4} P_\kappa \\ &= 2(\cos^2\alpha - \sin^2\alpha) \sin\alpha \cos\alpha P_\kappa \;. \end{aligned} \tag{6.74}$$

Now, $2(\cos^2\alpha - \sin^2\alpha)\sin\alpha\cos\alpha = \cos 2\alpha \sin 2\alpha = \cos 2\alpha \sin 2\alpha = 1/2 \sin 4\alpha$, and this factor makes the correlation function vanish, thus

$$\xi_{tx}(\phi) = 0 \;. \tag{6.75}$$

It therefore makes sense to define the correlation functions

$$\xi_\pm(\phi) \equiv \langle \gamma_t \gamma_t' \rangle \pm \langle \gamma_x \gamma_x' \rangle \;, \tag{6.76}$$

and

$$\xi_x(\phi) \equiv \langle \gamma_t \gamma_x' \rangle \;. \tag{6.77}$$

such that

$$\xi_+(\phi) = \int \frac{l\mathrm{d}l}{2\pi} P_\kappa(l) J_0(l\phi) \;, \tag{6.78}$$

$$\xi_-(\phi) = \int \frac{l\mathrm{d}l}{2\pi} P_\kappa(l) J_4(l\phi) \;, \tag{6.79}$$

and the expectation value of $\xi_(\phi) = 0$. For any measurement of cosmic shear, $\xi_x(\phi) = 0$ provides a test for the reliability of the measurement, because $\xi_x(\phi) \neq 0$ points at systematic errors.

6.5 Shear in apertures and aperture mass

Another convenient measure for the magnitude of the shear is to compute the mean shear in a (circular) aperture of radius θ,

$$\gamma_{\rm av}(\theta) = \int_0^\theta \frac{{\rm d}^2\Theta}{\pi\Theta^2}\gamma(\vec\Theta)\;, \tag{6.80}$$

and to study its variance,

$$\begin{aligned}\langle|\gamma_{\rm av}|^2\rangle &= \left\langle\int_0^\theta \frac{{\rm d}^2\Theta}{\pi\Theta^2}\int_0^\theta \frac{{\rm d}^2\Theta'}{\pi\Theta'^2}[\gamma_1(\vec\Theta)\gamma_1(\vec\Theta')+\gamma_2(\vec\Theta)\gamma_2(\vec\Theta')]\right\rangle \\ &= \int_0^\theta \frac{{\rm d}^2\Theta}{\pi\Theta^2}\int_0^\theta \frac{{\rm d}^2\Theta'}{\pi\Theta'^2}\xi_\kappa(|\vec\Theta'-\vec\Theta|)\;. \end{aligned}\tag{6.81}$$

The latter equality follows from the fact that the correlation functions of convergence and absolute value of the sheare are identical, as is best seen from their power spectra. We have

$$\begin{aligned}P_\gamma &= \left[\left(\frac{k_1^2-k_2^2}{2}\right)^2 + k_1^2k_2^2\right]P_\psi \\ &= \frac{1}{4}(k_1^4+k_2^4-2k_1^2k_2^2+4k_1^2k_2^2)P_\psi \\ &= \frac{1}{4}(k_1^2+k_2^2)^2P_\psi = \frac{k^4}{4}P_\psi = P_\kappa\;. \end{aligned}\tag{6.82}$$

Inserting the convergence power spectrum into the shear variance, we obtain

$$\begin{aligned}\langle|\gamma_{\rm av}|^2\rangle(\theta) &= \int_0^\theta \frac{{\rm d}^2\Theta}{\pi\Theta^2}\frac{{\rm d}^2\Theta'}{\pi\Theta'^2}\int\frac{{\rm d}^2l}{(2\pi)^2}P_\kappa(l)\exp(-i\vec l(\vec\Theta-\vec\Theta') \\ &= 4\pi^2\int_0^\infty \frac{l{\rm d}l}{2\pi}P_\kappa(l)\left[\frac{J_1(l\Theta)}{\pi l\Theta}\right]^2\;, \end{aligned}\tag{6.83}$$

for which we have used

$$\int_0^1 x{\rm d}xJ_0(ax) = \frac{1}{a}J_1(a)\;. \tag{6.84}$$

The aperture mass is defined as a weighted integral over the (effective) convergence within a (circular) aperture,

$$M_{\rm ap}(\theta) = \int_0^\theta {\rm d}^2\Theta U(\vec\Theta)\kappa_{\rm eff}(\vec\Theta)\;. \tag{6.85}$$

If the weight function satisfies the condition

$$\int_0^\theta \Theta{\rm d}\Theta U(\Theta) = 0\;, \tag{6.86}$$

i.e. if it is compensated, the aperture mass can also be written as

$$M_{\rm ap}(\theta) = \int_0^\theta {\rm d}^2\Theta Q(\Theta)\gamma_t(\vec\Theta)\;, \tag{6.87}$$

where γ_t is the tangential shear with respect to the aperture centre. Q is related to U through

$$Q(x) = \frac{2}{x^2}\int_0^x \mathrm{d}x' x' U(x') - U(x)\ . \tag{6.88}$$

A common choice (but not a neccessary one) is

$$U(\Theta) = \frac{9}{\pi\Theta^2}(1-x^2)\left(\frac{1}{3}-x^2\right)\ , \tag{6.89}$$

with $x \equiv \theta/\Theta$. This implies

$$Q(\Theta) = \frac{6}{\pi\Theta^2}x^2(1-x^2)\ . \tag{6.90}$$

Using this choice the variance of the aperture mass turns out to be

$$\begin{aligned}
\langle M_{\rm ap}^2\rangle &= \left\langle \int_0^\theta \mathrm{d}^2\Theta \int_0^\theta \mathrm{d}^2\Theta' U(\Theta)U(\Theta')\kappa_{\rm eff}(\vec{\Theta})\kappa_{\rm eff}(\vec{\Theta}')\right\rangle \\
&= \int \mathrm{d}^2\Theta \int \mathrm{d}^2\Theta' U(\Theta)U(\Theta')\xi_\kappa(|\vec{\Theta}'-\vec{\Theta}|) \\
&= \int \mathrm{d}^2\Theta \int \mathrm{d}^2\Theta' U(\Theta)U(\Theta')\int \frac{\mathrm{d}^2 l}{(2\pi)^2}P_\kappa(l)\exp[-i\vec{l}(\vec{\Theta}-\vec{\Theta}')] \\
&= 4\pi\int \frac{l\mathrm{d}l}{2\pi}P_\kappa(l)J^2(l\theta)\ ,
\end{aligned} \tag{6.91}$$

where

$$J(l\theta) \equiv \frac{12}{\pi(l\theta)^2}J_4(l\theta)\ . \tag{6.92}$$

Obviously, the magnification correlation, the shear correlation functions $\xi_p m$, the shear in apertures or the aperture mass all measure weighted integrals of $P_\kappa(l)$, where the weight functions can be more or less peaked.

6.6 E- and B-modes

Shear caused by gravitational lensing cannot have a curl component because of its origin in a scalar potential. If there is a curl component in measured shear, it must therefore be caused by systematic effects. In analogy to electromagnetic fields, real (scalar) modes are called E-modes, others (vectorial modes) are called B-modes.

Assuming E- and B-modes to be independent, the shear powerspectrum can be written as

$$\langle\gamma(\vec{l})\gamma^*(\vec{l}')\rangle = (2\pi)^2\delta_D^2(\vec{l}-\vec{l}')[P_E(l)+P_B(l)]\ . \tag{6.93}$$

As before, we now use

$$\begin{aligned}
\gamma_t &= \gamma(\cos^2\alpha - \sin^2\alpha) = \gamma\cos 2\alpha & (6.94)\\
\gamma_x &= \gamma 2\sin\alpha\cos\alpha = \gamma\sin 2\alpha\ , & (6.95)
\end{aligned}$$

and find

$$\xi_+(\phi) = \int \frac{l\mathrm{d}l}{2\pi}[P_E(l)+P_B(l)]J_0(l\phi) \tag{6.96}$$

and

$$\xi_-(\phi) = \int \frac{l\mathrm{d}l}{2\pi}[P_E(l) - P_B(l)]J_4(l\phi) \; . \tag{6.97}$$

Using Fourier transforms, these relations can be inverted to yield

$$P_E(l) = \int_0^\infty \phi\mathrm{d}\phi[\xi_+(\phi)J_0(l\phi) + \xi_-(\phi)J_4(l\phi)] \; , \tag{6.98}$$

$$P_E(l) = \int_0^\infty \phi\mathrm{d}\phi[\xi_+(\phi)J_0(l\phi) - \xi_-(\phi)J_4(l\phi)] \; . \tag{6.99}$$

This allows quantifying E- and B-mode contributions to the signal.
In particular, the aperture mass is only sensitive to E modes, while

$$M_{\mathrm{ap}\perp} \equiv \int_0^\theta \Theta\mathrm{d}\Theta Q(\Theta)\gamma_x(\Theta) \tag{6.100}$$

is only sensitive to B-modes. That way, they can be easily compared.

6.7 Lensing of the Cosmic Microwave Background

Lensing also changes the appearance of the CMB, because temperature fluctuations originally at a position $\vec{\beta}$ are shifted to $\vec{\theta} = \vec{\beta} + \vec{\alpha}$ due to lensing.
The CMB is characterized by its relative temperature fluctuations

$$\tau(\vec{\theta}) \equiv \frac{T(\vec{\theta})}{\langle T \rangle} \tag{6.101}$$

and their power spectrum

$$P_T(l) = \langle \hat{\tau}(\vec{l})\hat{\tau}^*(\vec{l}') \rangle = \int \mathrm{d}^2\phi \xi_T(\phi) \exp(-i\vec{\phi}\vec{l}) \; . \tag{6.102}$$

We wish to calculate how the power spectrum of the CMB will change due to lensing.
The temperature autocorrelation function without lensing would be

$$\langle \tau(\vec{\theta})\tau(\vec{\theta} + \vec{\phi}) \rangle \; , \tag{6.103}$$

with lensing it is

$$\langle \tau(\vec{\theta} - \vec{\alpha})\tau(\vec{\theta}' - \vec{\alpha}') \rangle \; , \tag{6.104}$$

where $\vec{\alpha} = \vec{\alpha}(\vec{\theta})$ and $\vec{\alpha}' = \vec{\alpha}'(\vec{\theta}')$; $\vec{\theta}' = \vec{\theta} + \vec{\phi}$.
In terms of Fourier transforms

$$\tau(\vec{\theta} - \vec{\alpha}) = \int \frac{\mathrm{d}^2 l}{(2\pi)^2}\hat{\tau}(\vec{l}) \exp[-i(\vec{\theta} - \vec{\alpha})] \; , \tag{6.105}$$

thus

$$\begin{aligned}
\xi_T(\phi) &= \left\langle \int \frac{\mathrm{d}^2 l}{(2\pi)^2}\hat{\tau}(\vec{l}) \exp[-i(\vec{\theta} - \vec{\alpha})\vec{l}] \int \frac{\mathrm{d}^2 l'}{(2\pi)^2}\hat{\tau}^*(\vec{l}') \exp[i(\vec{\theta}' - \vec{\alpha}')\vec{l}'] \right\rangle \\
&= \int \frac{\mathrm{d}^2 l}{(2\pi)^2} \int \frac{\mathrm{d}^2 l'}{(2\pi)^2} \langle \hat{\tau}(\vec{l})\hat{\tau}^*(\vec{l}') \rangle \langle \exp[-i(\vec{\theta} - \vec{\alpha})\vec{l}] \exp[i(\vec{\theta}' - \vec{\alpha}')\vec{l}'] \rangle \\
&= \int \frac{\mathrm{d}^2 l}{(2\pi)^2} P_T(l) \exp[-i(\vec{\theta} - \vec{\theta}')\vec{l}] \langle \exp[i(\vec{\alpha} - \vec{\alpha}')\vec{l} \rangle \; .
\end{aligned} \tag{6.106}$$