GRAVITATIONAL LENSING

LECTURE 2

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AA 2015-2016

CONTENTS

- Lens equation
- Lensing potential

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

From last lecture

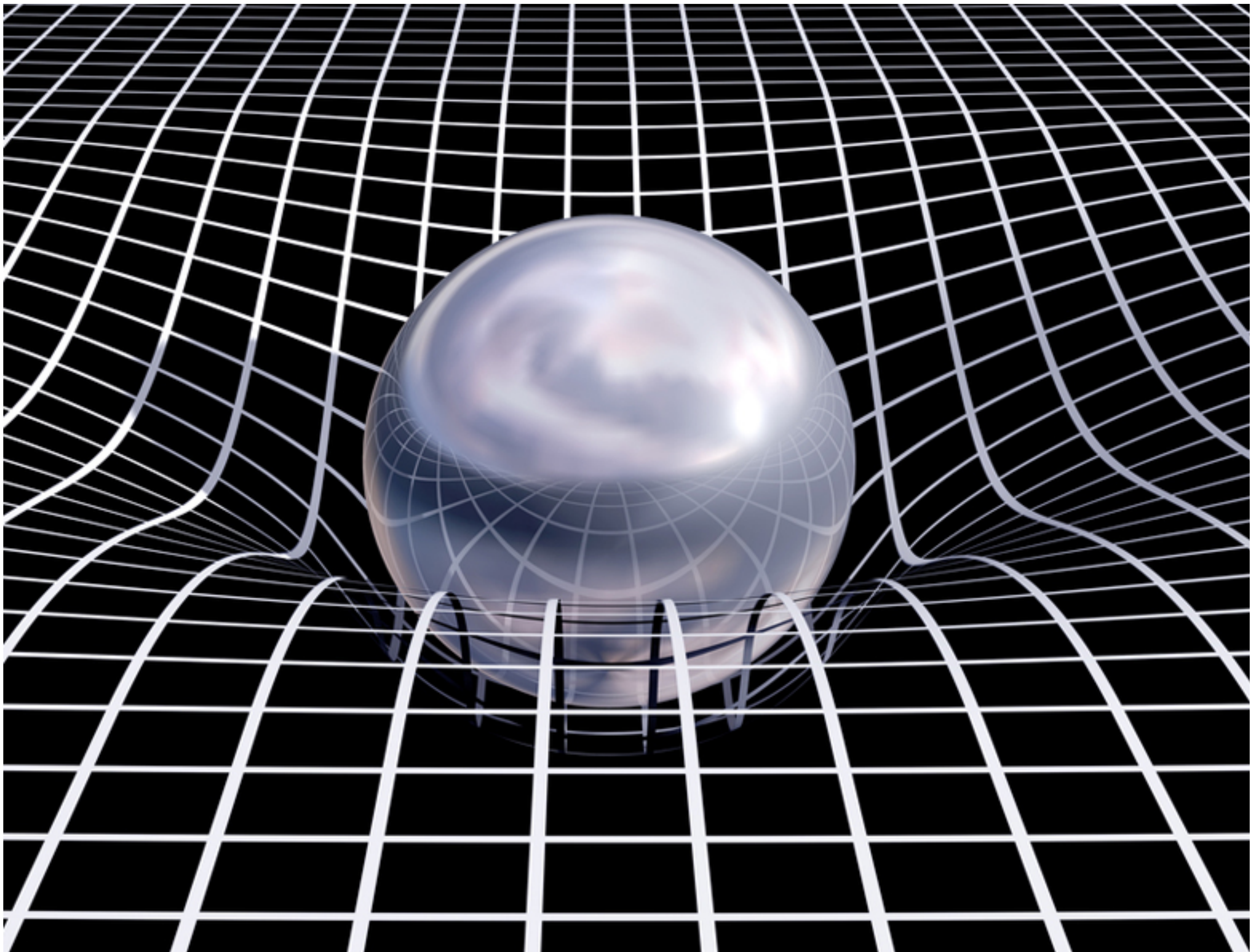
How to define the effective diffraction index?

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 = \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \approx c\left(1 + \frac{2\Phi}{c^2}\right)$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY



SCHWARZSCHILD METRIC

$$ds^2 = \left(1 - \frac{2GM}{Rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{Rc^2}\right)^{-1} dR^2 - R^2(\sin^2\theta d\phi^2 + d\theta^2)$$

$$R = \sqrt{1 + \frac{2GM}{rc^2}}\, r$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\phi$$

$$dl^2 = [dr^2 + r^2(\sin^2\theta d\phi^2 + d\theta^2)]$$

In the weak field limit:

$$\begin{aligned} \left(1 - \frac{2GM}{Rc^2}\right) &= 1 - \frac{2GM}{c^2 r} \frac{1}{\sqrt{1 + \frac{2GM}{c^2 r}}} \\ &\approx 1 - \frac{2GM}{c^2 r}\left(1 - \frac{GM}{c^2 r}\right) \\ &\approx 1 - \frac{2GM}{c^2 r} \end{aligned}$$

$$\begin{aligned} \left(1 - \frac{2GM}{Rc^2}\right)^{-1} &\approx 1 + \frac{2GM}{c^2 R} \\ &= 1 + \frac{2GM}{c^2 r} \frac{1}{\sqrt{1 + \frac{2GM}{c^2 r}}} \\ &\approx 1 + \frac{2GM}{c^2 r}\left(1 - \frac{GM}{c^2 r}\right) \\ &\approx 1 + \frac{2GM}{c^2 r} \end{aligned}$$

SCHWARZSCHILD METRIC IN THE WEAK FIELD LIMIT

$$ds^2 = \left(1 - \frac{2GM}{Rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{Rc^2}\right)^{-1} dR^2 - R^2(\sin^2\theta d\phi^2 + d\theta^2)$$

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 + \frac{2GM}{rc^2}\right) dl^2$$

$$\Phi = -\frac{GM}{r}$$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

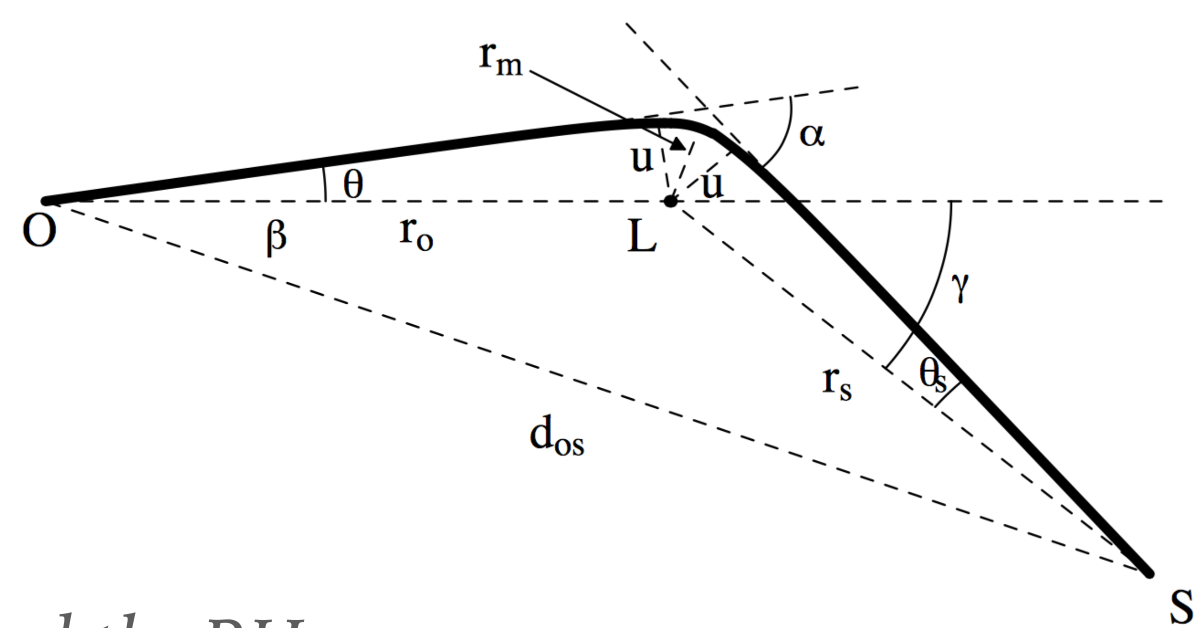
DEFLECTION OF LIGHT BY A BLACK HOLE

suggested reading: http://arxiv.org/pdf/0911.2187v2.pdf

Generic static spherically symmetric metric:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\vartheta^2 + \sin^2\vartheta d\phi^2).$$

$$\alpha = -\pi + 2\int_{r_m}^{\infty} u\sqrt{\frac{B(r)}{C(r)\left[C(r)/A(r) - u^2\right]}}dr,$$

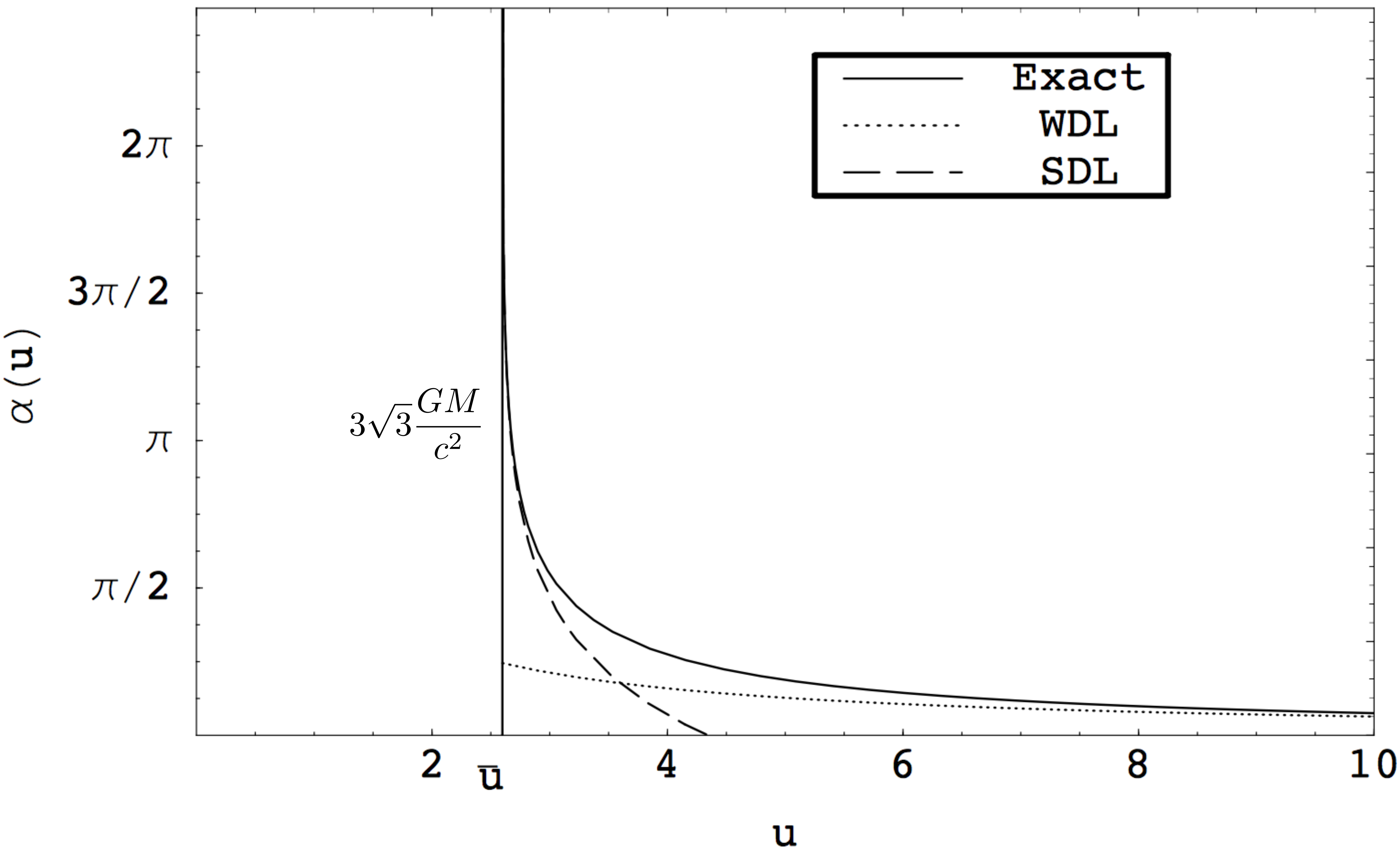


u=impact parameter

r_m=minimum distance between the photon and the BH

DEFLECTION OF LIGHT BY A BLACK HOLE

for the Schwarzschild metric

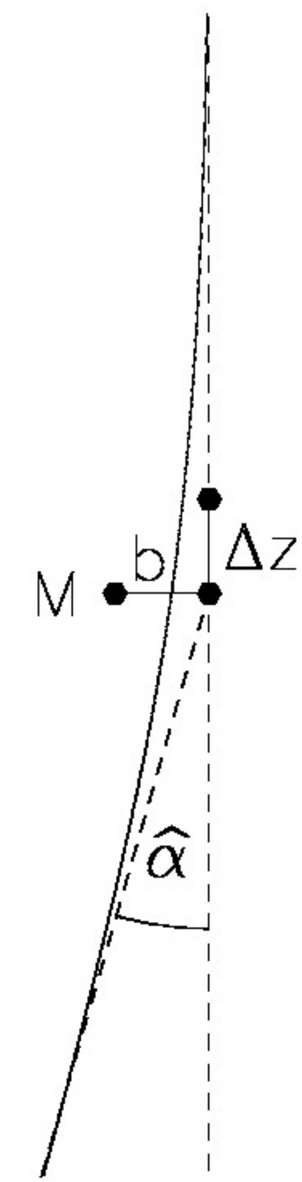


DEFLECTION ANGLE OF A POINT MASS

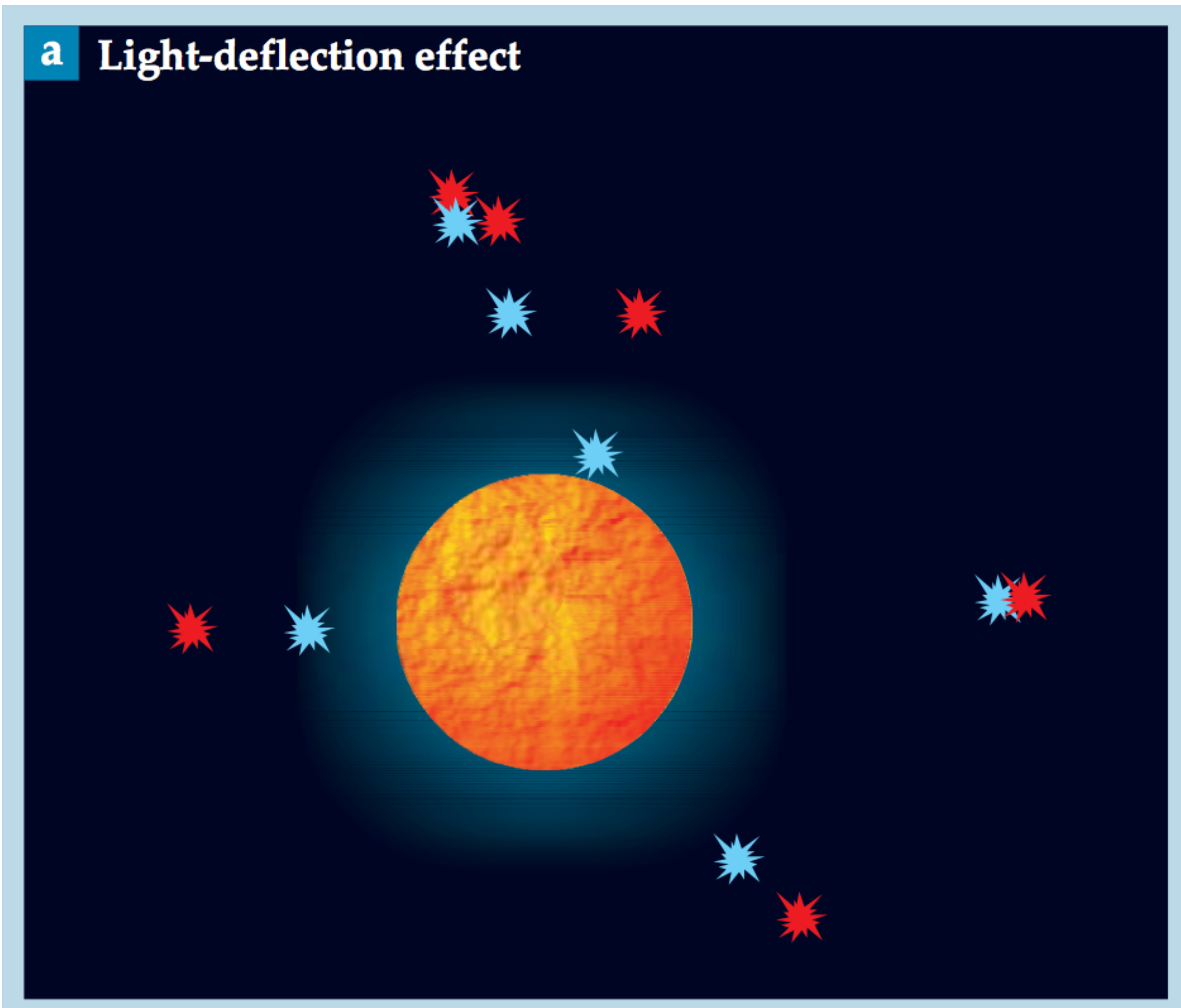
$$\hat{\vec{\alpha}} = \frac{2}{c^2} \int_{-\infty}^{+\infty} \vec{\nabla}_\perp \Phi dz$$

$$\Phi = -\frac{GM}{r}$$

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{dz}{(b^2 + z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2 (b^2 + z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \end{aligned}$$

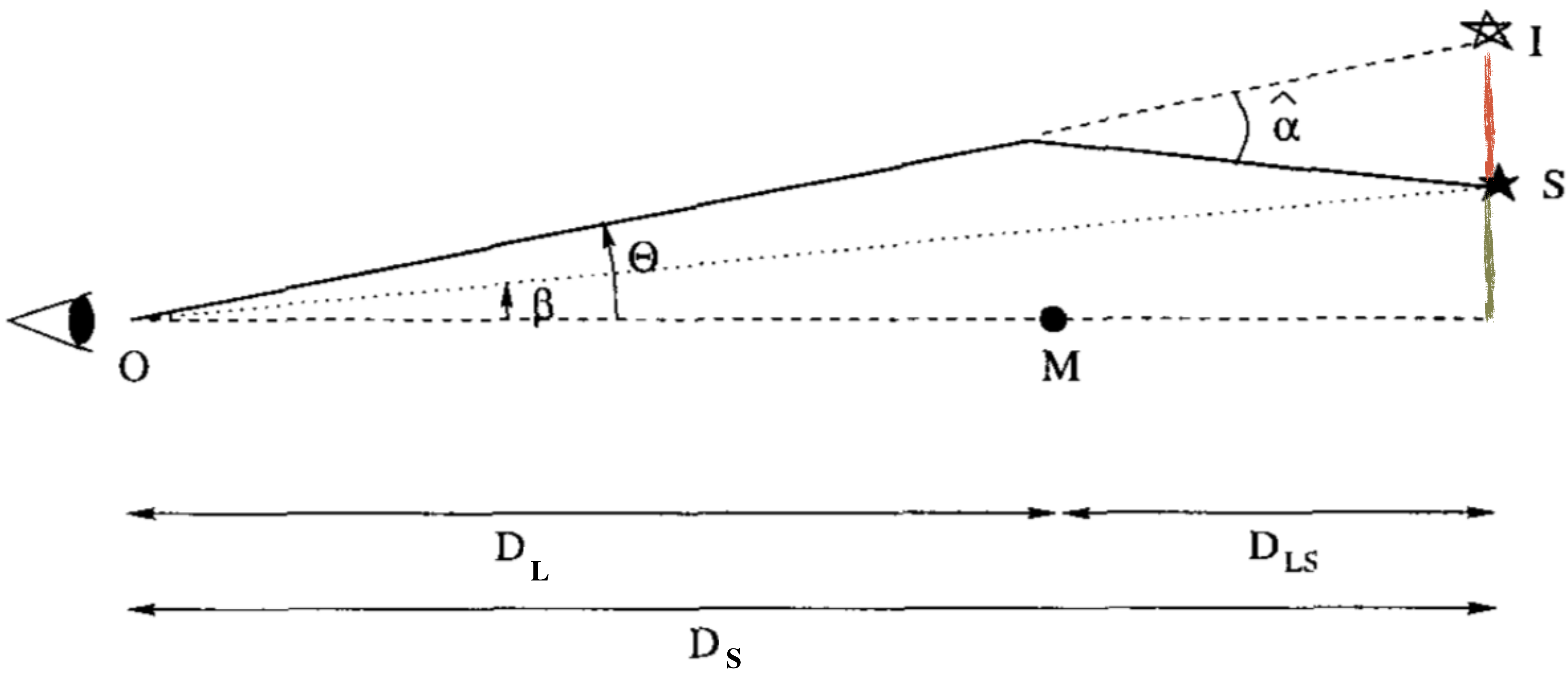


AGAIN ON THE EDDINGTON EXPEDITION



- The goal of Eddington expeditions was to measure a shift in the position of the Hyades stars due to solar deflection
- What is the exact shift we should expect to measure?
- What is the relation between the intrinsic and the apparent positions of the stars?

LENS EQUATION



COMOVING DISTANCE

suggested reading: http://arxiv.org/pdf/astro-ph/9905116v4.pdf

comoving distance *(along the line of sight) = distance between two points which remains constant if the two points are moving with the Hubble flow*

proper distance *= distance between the two points measured by rulers at the time they are being observed*

$$D_c = D_{pr}(1+z)$$

$$E(z) \equiv \sqrt{\Omega_{\rm M}\,(1+z)^3 + \Omega_k\,(1+z)^2 + \Omega_\Lambda}$$

$$\Omega_{\rm M} \equiv \frac{8\pi\, G\, \rho_0}{3\, H_0^2}$$

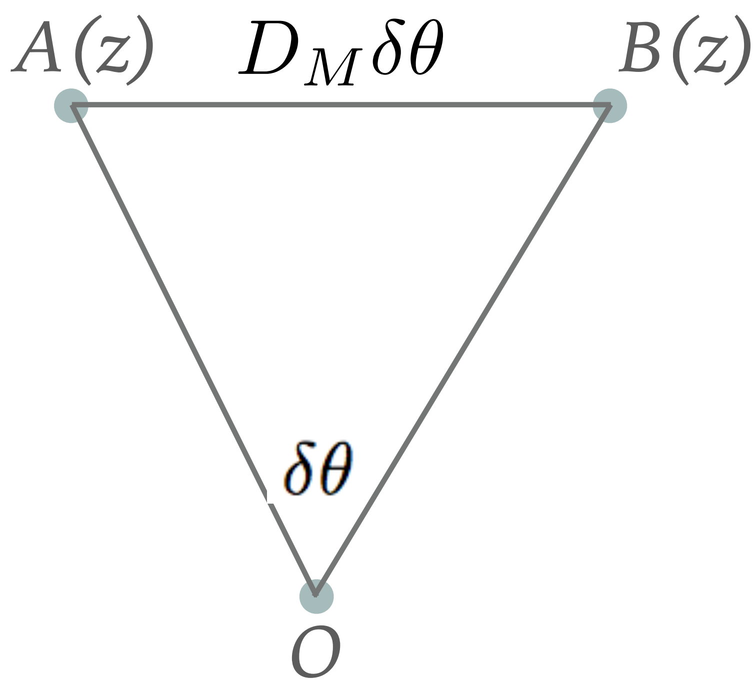
$$D_{\rm C} = D_{\rm H} \int_0^z \frac{dz'}{E(z')}$$

$$\Omega_\Lambda \equiv \frac{\Lambda\, c^2}{3\, H_0^2}$$

$$\Omega_{\rm M} + \Omega_\Lambda + \Omega_k = 1$$

$$D_{\rm H} \equiv \frac{c}{H_0} = 3000\, h^{-1}\ {\rm Mpc} = 9.26 \times 10^{25}\, h^{-1}\ {\rm m}$$

ANGULAR DIAMETER DISTANCE



$D_M \delta\theta$ = *comoving transversal distance*

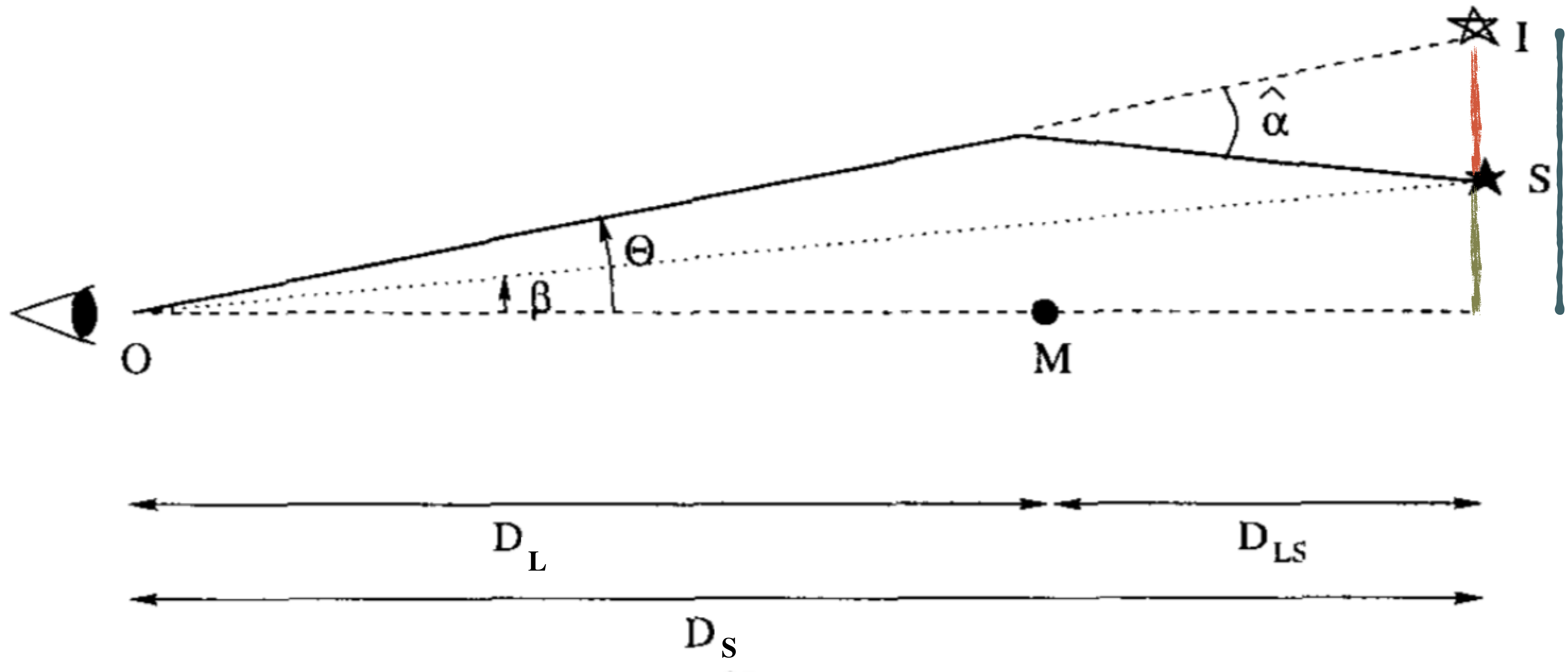
$$D_{\rm M} = \begin{cases} D_{\rm H} \frac{1}{\sqrt{\Omega_k}} \sinh\left[\sqrt{\Omega_k}\, D_{\rm C}/D_{\rm H}\right] & \text{for } \Omega_k > 0 \\ D_{\rm C} & \text{for } \Omega_k = 0 \\ D_{\rm H} \frac{1}{\sqrt{|\Omega_k|}} \sin\left[\sqrt{|\Omega_k|}\, D_{\rm C}/D_{\rm H}\right] & \text{for } \Omega_k < 0 \end{cases}$$

$$D_{\rm A} = \frac{D_{\rm M}}{1+z}$$

= *angular diameter distance*

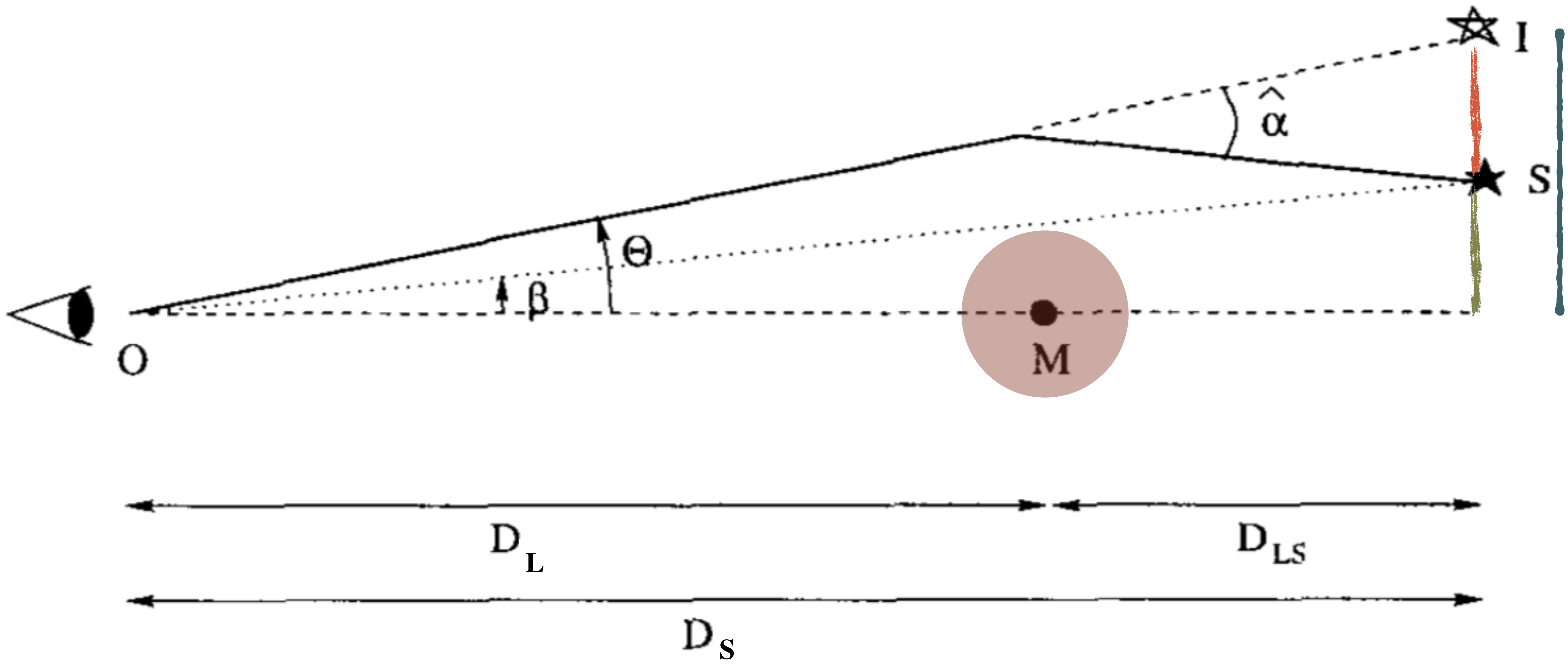
= *ratio of the physical (proper) transverse size to its angular size*

LENS EQUATION



$$\beta = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}$$

LENS EQUATION



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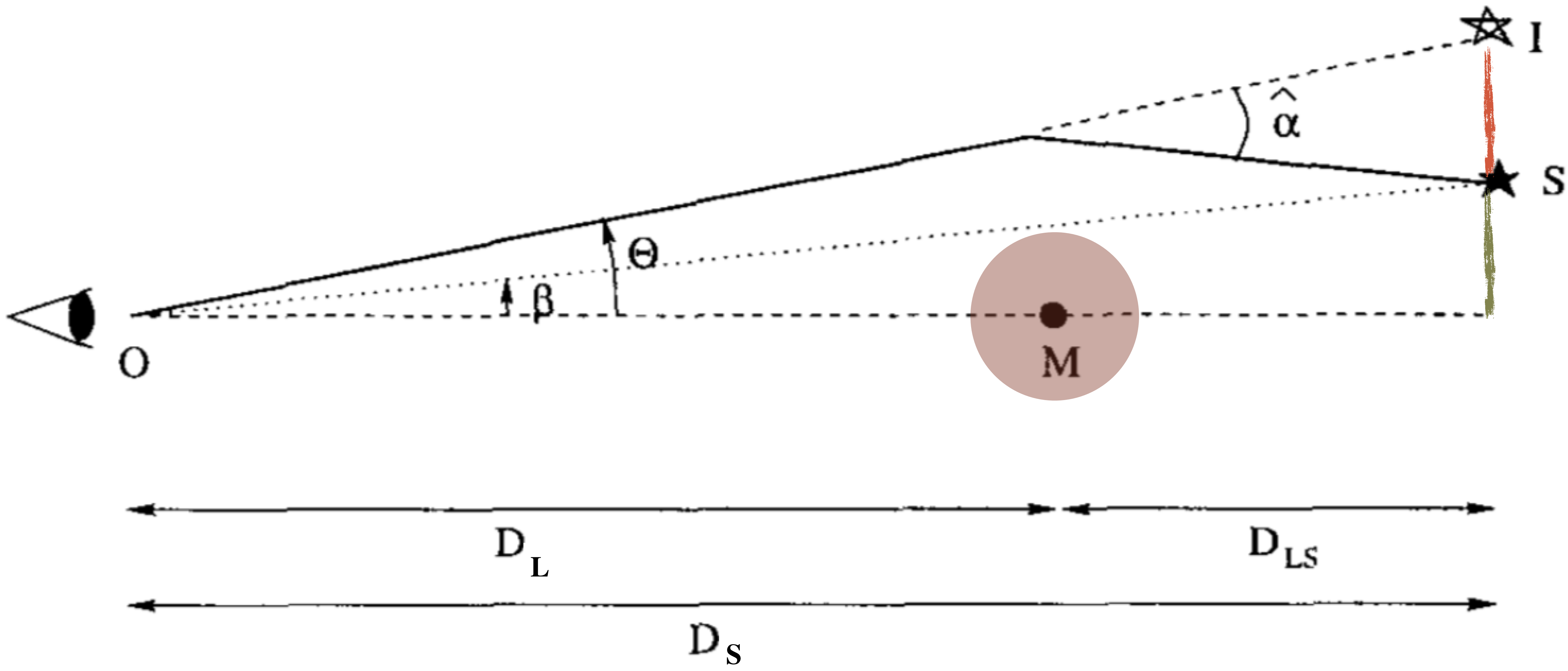

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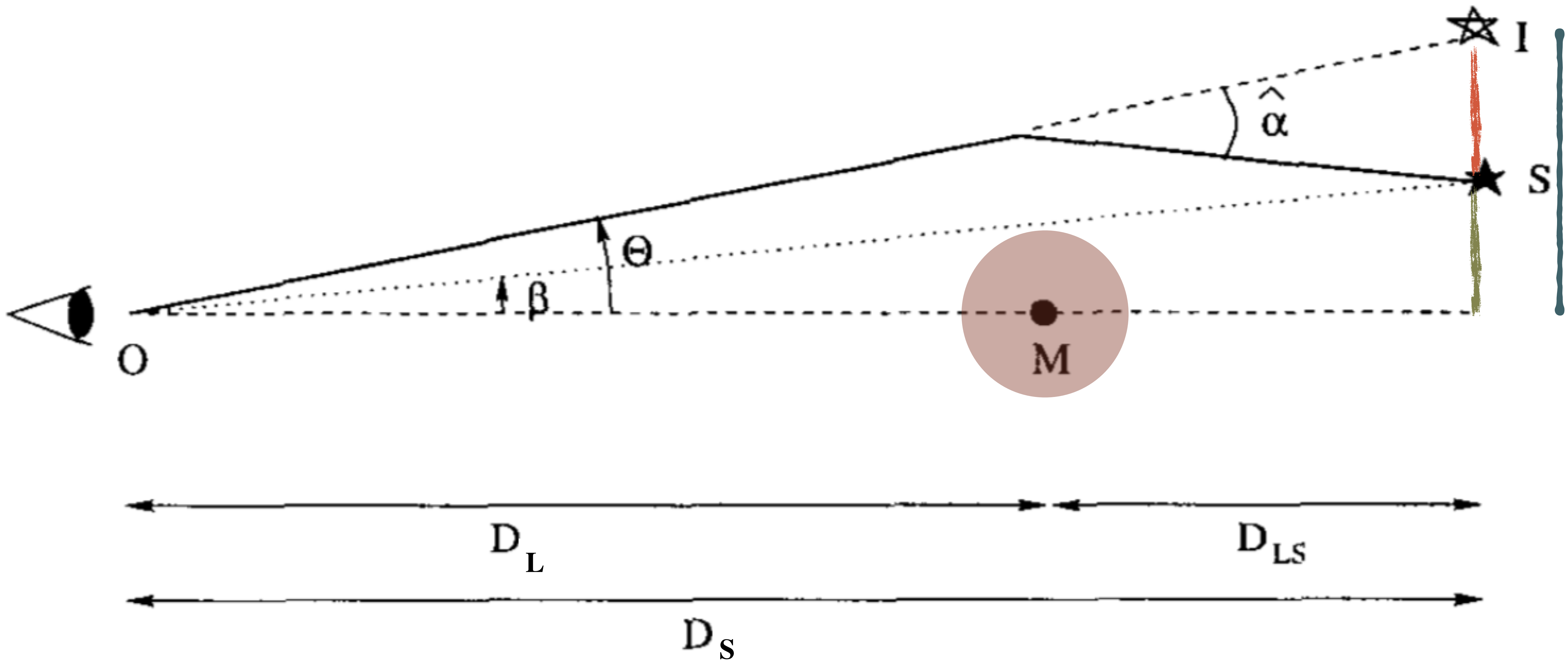

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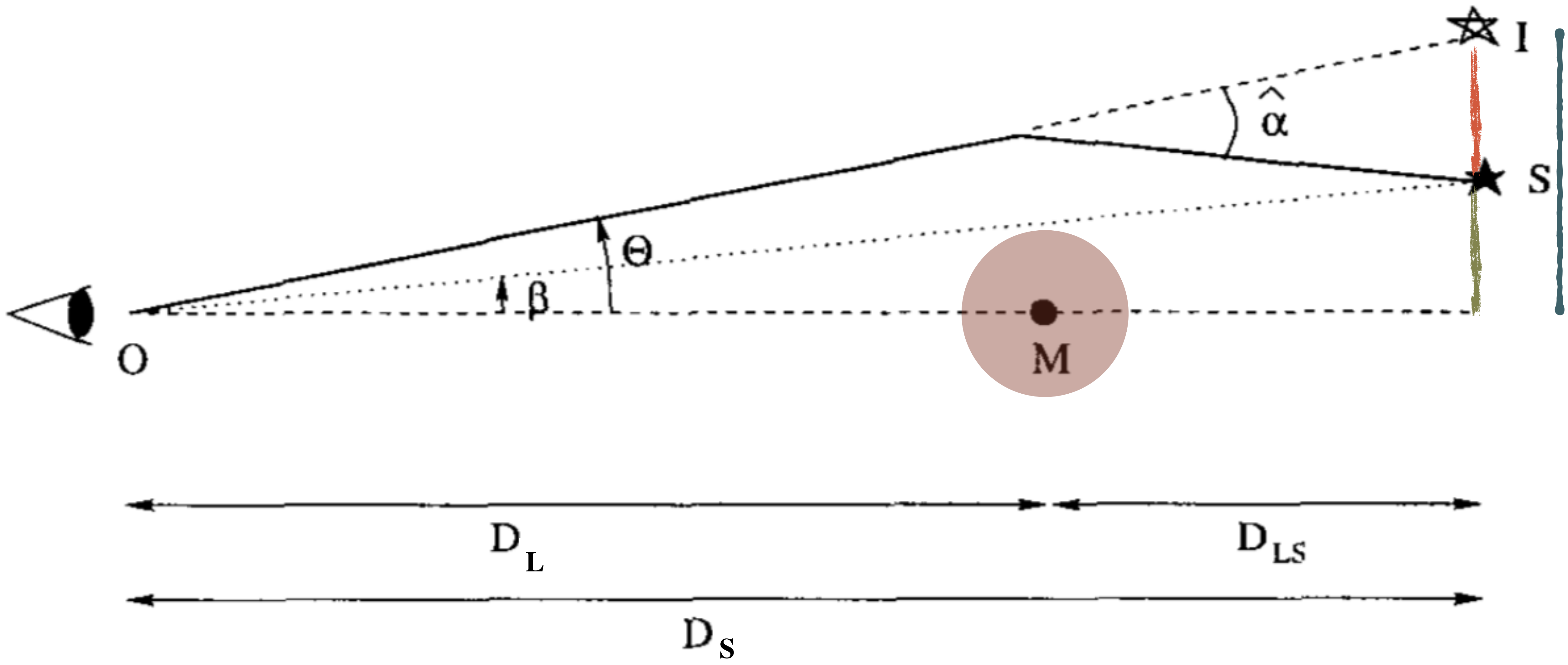
LENS EQUATION

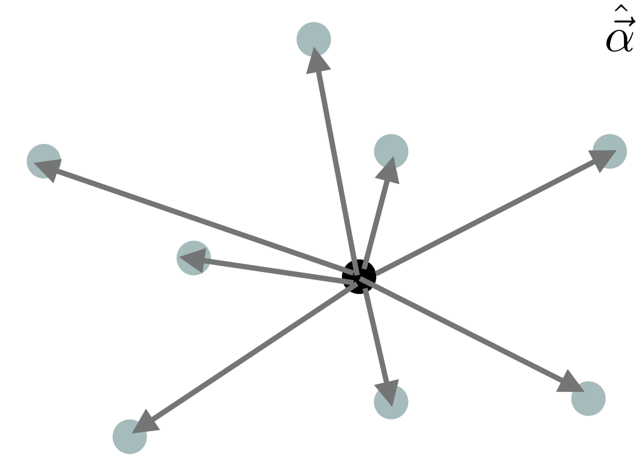


$$\beta = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}$$

DEFLECTION BY EXTENDED LENSES

- Remaining in the weak field limit, one can use the superposition principle
- The deflection angle by a system of point masses is the vectorial sum of the deflection angles of the single lenses
- This can be easily generalized to the case of a continuum distribution of mass
- Assumption: thin screen approximation

$$\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \Phi dz$$



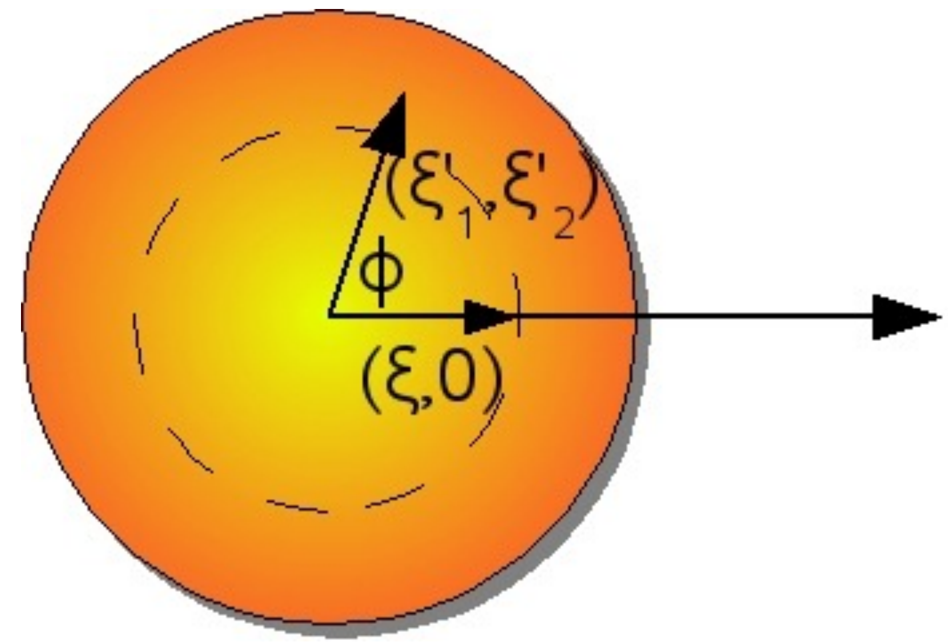
$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2}\sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\ \mathrm{d}z$$

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2}\int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}\ \mathrm{d}^2\xi'$$

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} \, \mathrm{d}^2\xi'$$



$$\begin{aligned} \vec{\xi} - \vec{\xi'} &= (\xi - \xi' \cos\phi, -\xi' \sin\phi) \\ |\vec{\xi} - \vec{\xi'}|^2 &= \xi^2 + \xi'^2 \cos^2\phi - 2\xi\xi' \cos\phi + \xi'^2 \sin^2\phi \\ &= \xi^2 + \xi'^2 - 2\xi\xi' \cos\phi \end{aligned}$$

$$\begin{aligned} \hat{\alpha}_1(\vec{\xi}) &= \frac{4G}{c^2} \int_0^\infty \mathrm{d}\xi' \xi' \Sigma(\xi') \int_0^{2\pi} \mathrm{d}\phi \frac{\xi - \xi' \cos\phi}{\xi^2 + \xi'^2 - 2\xi\xi' \cos\phi} \\ \hat{\alpha}_2(\vec{\xi}) &= \frac{4G}{c^2} \int_0^\infty \mathrm{d}\xi' \xi' \Sigma(\xi') \int_0^{2\pi} \mathrm{d}\phi \frac{-\xi' \sin\phi}{\xi^2 + \xi'^2 - 2\xi\xi' \cos\phi} \end{aligned}$$

$$\hat{\alpha}(\xi) = \frac{4G}{c^2} \frac{2\pi \int_0^\xi \Sigma(\xi')\xi' \, \mathrm{d}\xi'}{\xi} = \frac{4GM(\xi)}{c^2 \xi}$$

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

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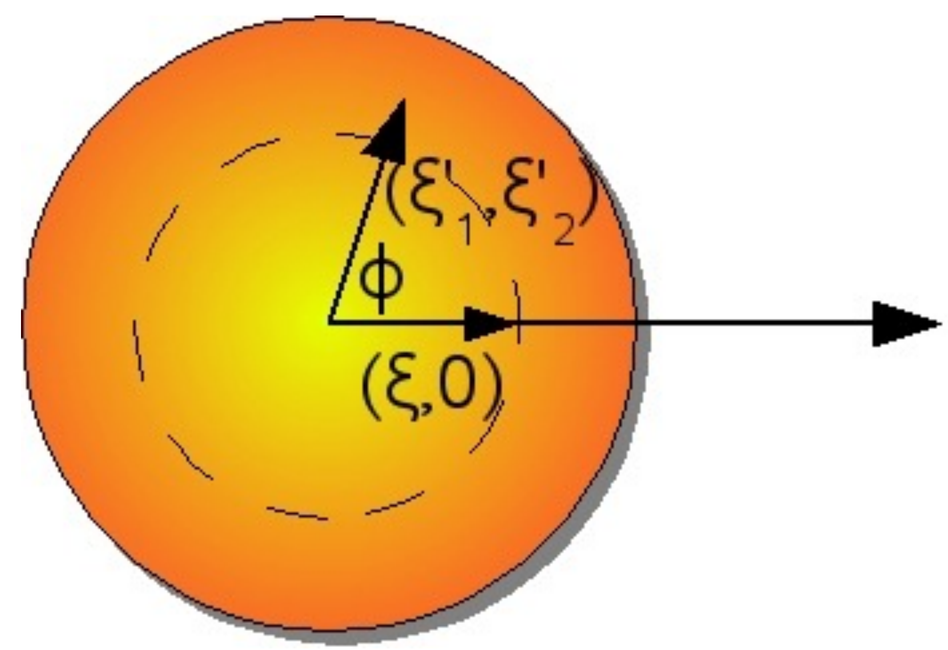

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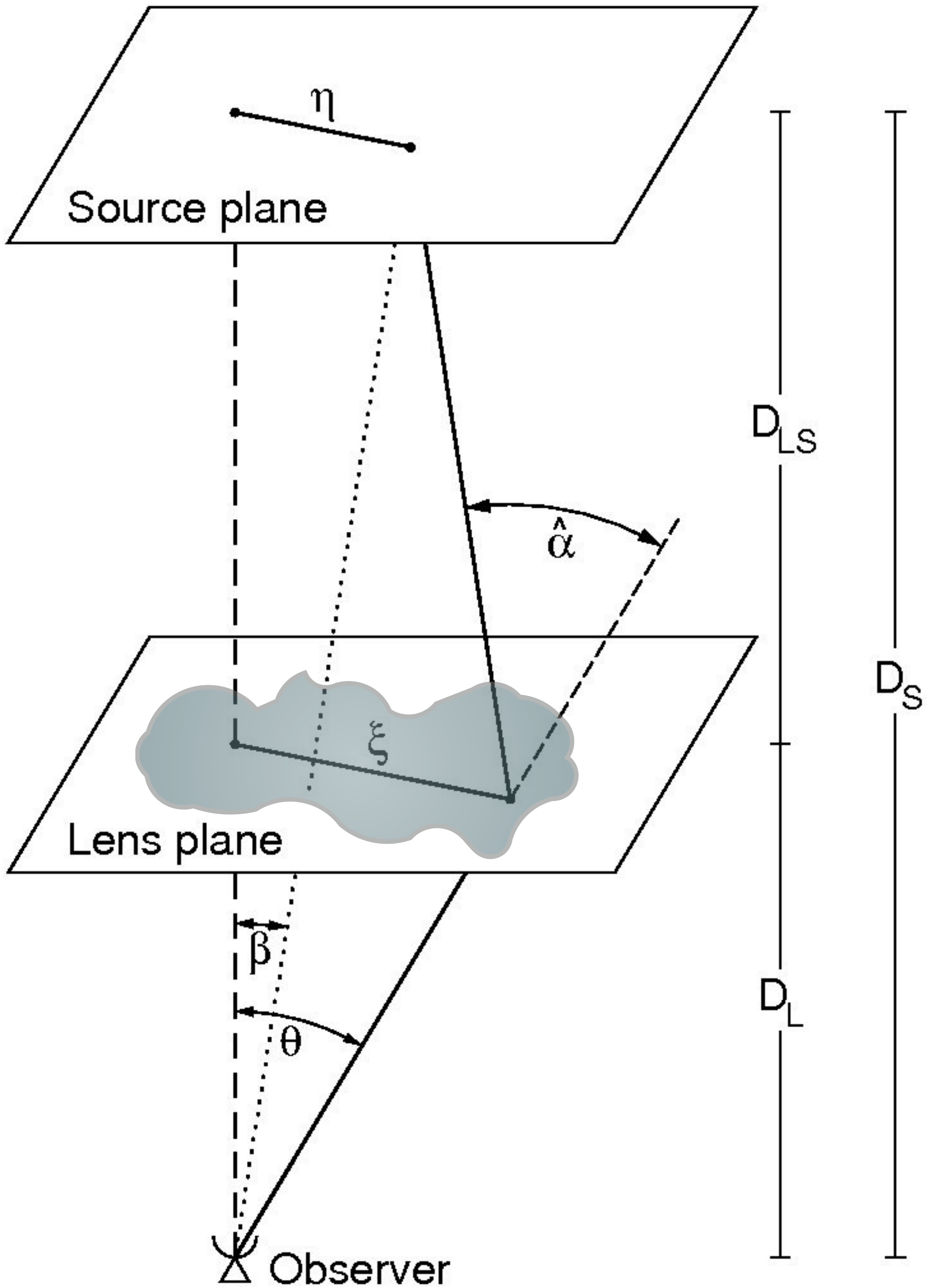
$$\frac{2\pi}{\xi}$$

$$\xi' < \xi$$

$$\hat{\alpha}(\xi) = \frac{4G}{c^2} \frac{2\pi \int_0^\xi \Sigma(\xi')\xi' \, \mathrm{d}\xi'}{\xi} = \frac{4GM(\xi)}{c^2 \xi}$$

LENS EQUATION



$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta})$$

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \qquad \vec{\beta} = \frac{\vec{\eta}}{D_S}$$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta})$$

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}$$

OTHER NOTATIONS

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \qquad \vec{\beta} = \frac{\vec{\eta}}{D_S} \qquad \vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) \qquad \vec{\beta} = \vec{\theta} - \vec{\alpha}$$

$$\theta_0 = \frac{\xi_0}{D_L} = \frac{\eta_0}{D_S}$$

$$\Downarrow$$

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \qquad \vec{\alpha}(\vec{x}) = \frac{\vec{\alpha}(\theta)}{\theta_0} = \frac{D_L}{\xi_0}\frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta})$$

LENSING POTENTIAL

$$\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \Phi dz$$

This formula tells us that the deflection is caused by the projection of the Newtonian gravitational potential on the lens plane.

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z) \mathrm{d}z$$

We introduce the ***effective lensing potential***

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*We introduce the **effective lensing potential***

1 *the lensing potential is the projection of the 3D potential*

2 *the lensing potential scales with distances*

OTHER PROPERTIES OF THE LENSING POTENTIAL

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

The deflection angle is the gradient of the lensing potential

$$\triangle_x \Psi(\vec{x}) = 2\kappa(\vec{x})$$

The laplacian of the lensing potential is twice the ***convergence***

OTHER PROPERTIES OF THE LENSING POTENTIAL

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

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$$\triangle_x \Psi(\vec{x}) = 2\kappa(\vec{x})$$

The laplacian of the lensing potential is twice the ***convergence***

$$\kappa(\vec{x}) \equiv \frac{\Sigma(\vec{x})}{\Sigma_{\mathrm{cr}}} \quad \text{with} \quad \Sigma_{\mathrm{cr}} = \frac{c^2}{4\pi G}\frac{D_{\mathrm{S}}}{D_{\mathrm{L}} D_{\mathrm{LS}}}$$