GRAVITATIONAL LENSING

LECTURE 3

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CONTENTS

- Lensing potential
- Lens mapping (first order)
- Distortion and magnification

LENSING POTENTIAL

$$\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \Phi dz$$

This formula tells us that the deflection is caused by the projection of the Newtonian gravitational potential on the lens plane.

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathrm{LS}}}{D_{\mathrm{L}} D_{\mathrm{S}}} \frac{2}{c^2} \int \Phi(D_{\mathrm{L}}\vec{\theta}, z) \mathrm{d}z$$

We introduce the ***effective lensing potential***

1 *the lensing potential is the projection of the 3D potential*

2 *the lensing potential scales with distances*

OTHER PROPERTIES OF THE LENSING POTENTIAL

$$\vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) = \vec{\alpha}(\vec{\theta})$$

The deflection angle is the gradient of the lensing potential

$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

The laplacian of the lensing potential is twice the ***convergence***

ADIMENSIONAL NOTATION

$$\vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) = \vec{\alpha}(\vec{\theta}) \quad \Rightarrow \quad \vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

$$\vec{\nabla}_x = \frac{\xi_0}{D_\mathsf{L}} \vec{\nabla}_\theta$$

$$\vec{\nabla}_x \hat{\Psi} = \frac{\xi_0}{D_\mathsf{L}} \vec{\nabla}_\theta \hat{\Psi} = \frac{\xi_0}{D_\mathsf{L}} \vec{\alpha}$$

$$\frac{D_\mathsf{L}^2}{\xi_0^2} \vec{\nabla}_x \hat{\Psi} = \frac{D_\mathsf{L}}{\xi_0} \vec{\alpha}$$

$$\Psi = \frac{D_\mathsf{L}^2}{\xi_0^2} \hat{\Psi}$$

ADIMENSIONAL NOTATION

$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\triangle_x \Psi(\vec{x}) = 2\kappa(\vec{x})$$

$$\kappa(\theta) = \frac{1}{2}\triangle_\theta \hat{\Psi} = \frac{1}{2}\frac{\xi_0^2}{D_\mathsf{L}^2}\triangle_\theta \Psi$$

$$\triangle_\theta = D_\mathsf{L}^2 \triangle_\xi = \frac{D_\mathsf{L}^2}{\xi_0^2}\triangle_x$$

Adimensional

ADIMENSIONAL NOTATION

From

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} \, \mathrm{d}^2\xi'$$

we obtain

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \mathrm{d}^2x' \kappa(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$$

Using

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

$$\Psi(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \kappa(\vec{x}') \ln |\vec{x} - \vec{x}'| \mathrm{d}^2x'$$

ADIMENSIONAL NOTATION

From

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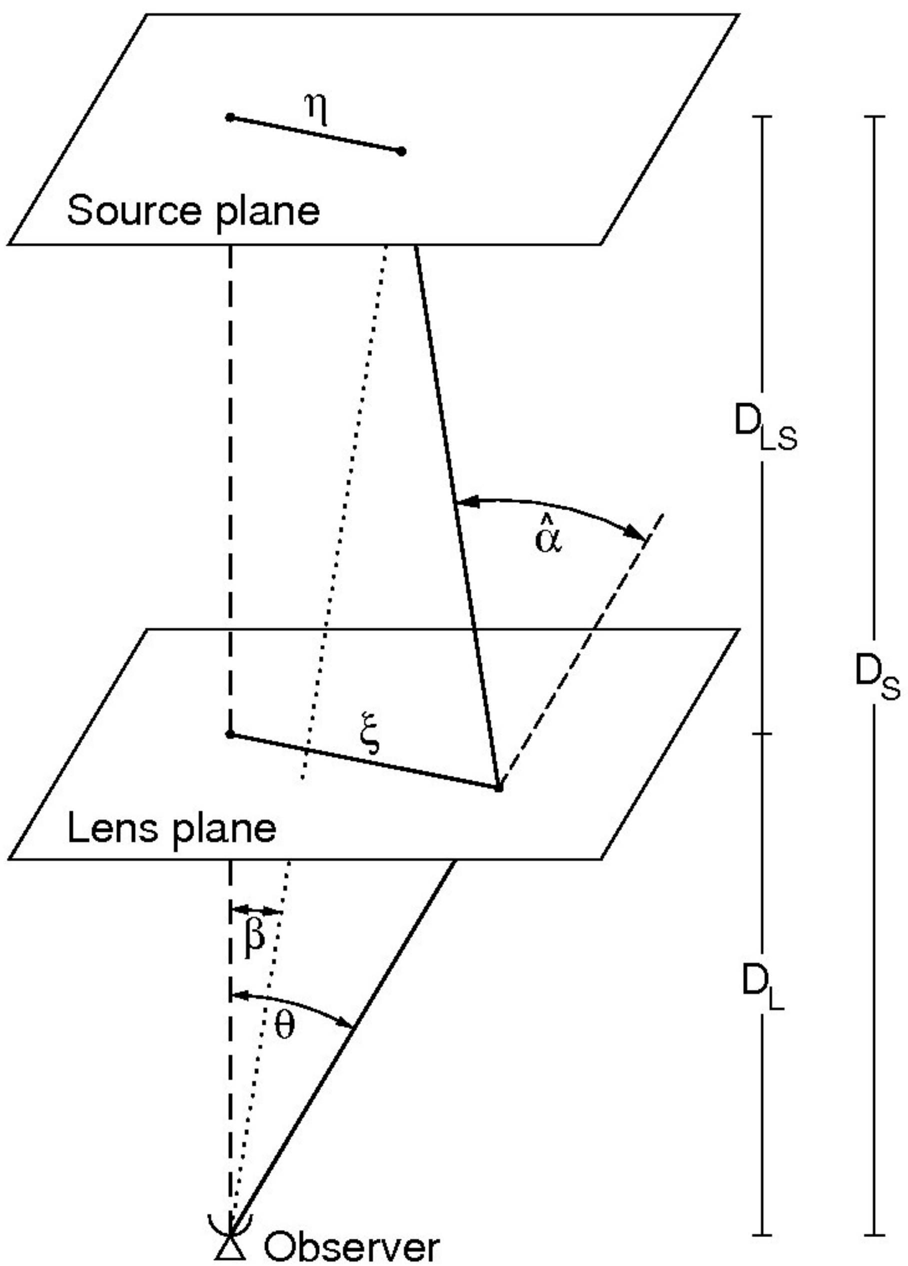
Using

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

Convolution kernels

$$\Psi(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \kappa(\vec{x}') \ln|\vec{x} - \vec{x}'| d^2x'$$

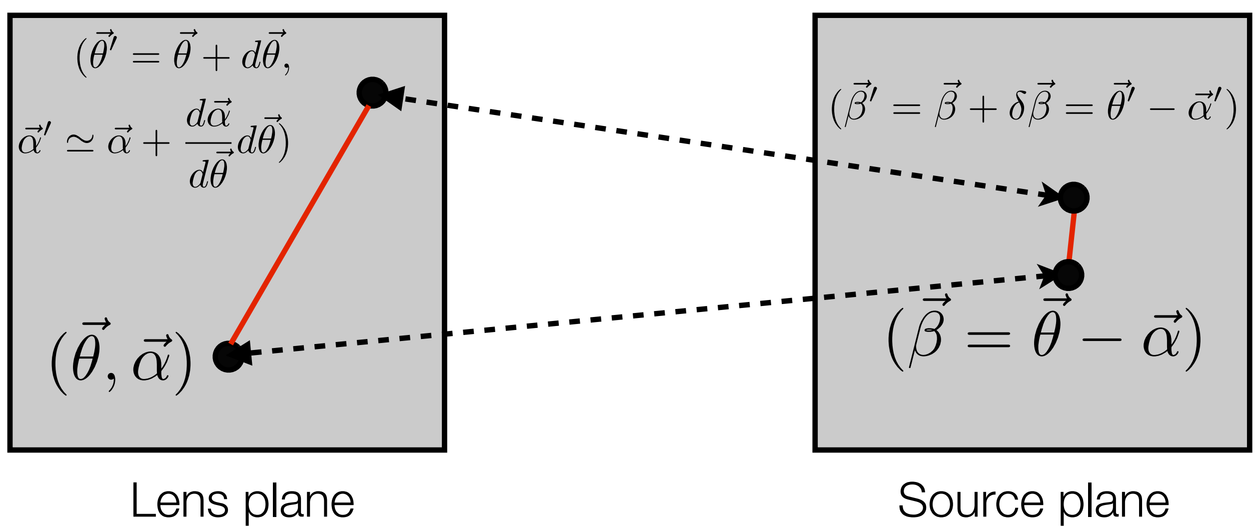
LENS MAPPING (FIRST ORDER)



- *Very simple equation to link points on the lens and on the source planes:*

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

- *Assuming that the d.a. does not vary significantly over the scale dΘ:*



$$(\vec{\beta}' - \vec{\beta}) = \left(I - \frac{d\vec{\alpha}}{d\vec{\theta}}\right)(\vec{\theta}' - \vec{\theta}) = A(\vec{\theta}' - \vec{\theta})$$

LENS MAPPING (FIRST ORDER)

$$A \equiv \frac{\partial \vec{y}}{\partial \vec{x}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{x})}{\partial x_j}\right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{x})}{\partial x_i \partial x_j}\right)$$

Symmetric second rank tensor describing the first order mapping between lens and source planes.

This tensor can be written as the sum of an isotropic part, proportional to its trace, and an anisotropic traceless part.

$$A_{iso,i,j} = \frac{1}{2}\mathrm{Tr}A\delta_{i,j}$$

$$A_{aniso,i,j} = A_{i,j} - \frac{1}{2}\mathrm{Tr}A\delta_{i,j}$$

ANISOTROPIC PART

$$A_{aniso,i,j} = A_{i,j} - \frac{1}{2}\mathrm{Tr}A\delta_{i,j}$$

$$\begin{aligned}\left(A - \frac{1}{2}\mathrm{tr}A \cdot I\right)_{ij} &= \delta_{ij} - \Psi_{ij} - \frac{1}{2}(1 - \Psi_{11} + 1 - \Psi_{22})\delta_{ij} \\ &= -\Psi_{ij} + \frac{1}{2}(\Psi_{11} + \Psi_{22})\delta_{ij} \\ &= \begin{pmatrix} -\frac{1}{2}(\Psi_{11} - \Psi_{22}) & -\Psi_{12} \\ -\Psi_{12} & \frac{1}{2}(\Psi_{11} - \Psi_{22}) \end{pmatrix}\end{aligned}$$

Introducing the ***shear****:*

$$\gamma_1 = \frac{1}{2}\left(\Psi_{11} - \Psi_{22}\right)$$

$$\gamma_2 = -\Psi_{12} = -\Psi_{21}$$

$$\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} = \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$$

Symmetric, trace-less tensor

ISOTROPIC PART

$$A_{iso,i,j} = \frac{1}{2}\mathrm{Tr}A\delta_{i,j}$$

$$\begin{aligned}\frac{1}{2}\mathrm{tr}A\cdot I &= \left[1-\frac{1}{2}(\Psi_{11}+\Psi_{22})\right]\delta_{ij} \\ &= \left(1-\frac{1}{2}\triangle\Psi\right)\delta_{ij} = (1-\kappa)\delta_{ij}\end{aligned}$$

Remember: $\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$

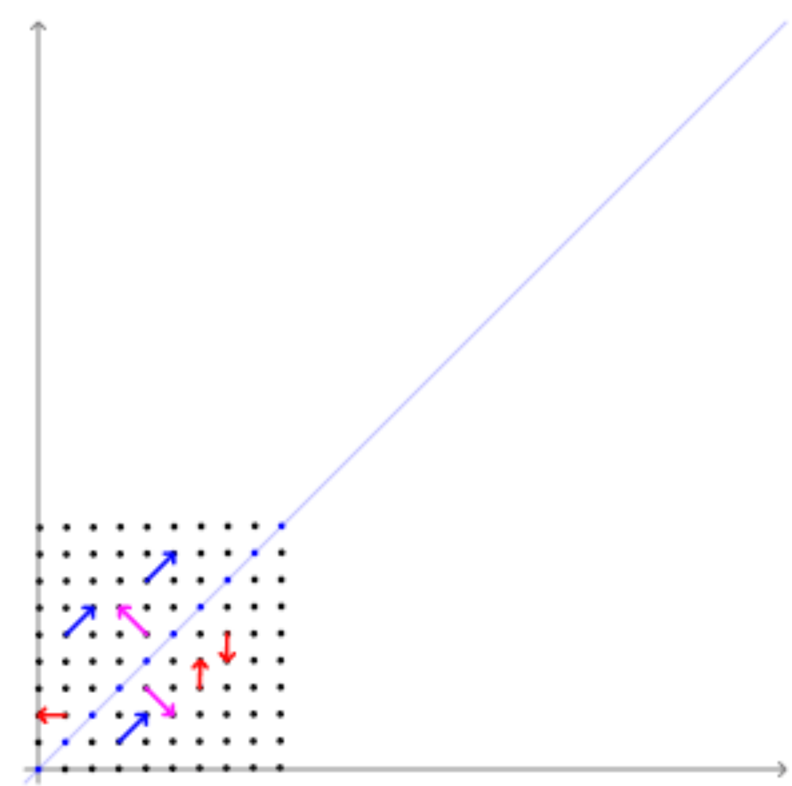
LENSING JACOBIAN

$$A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$
$$= (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$$

Lens mapping at first order can be seen as a linear application, distorting areas.

Distortion directions are given by the ***eigenvectors*** *of A.*

Distortion amplitudes in these directions are given by the ***eigenvalues****.*



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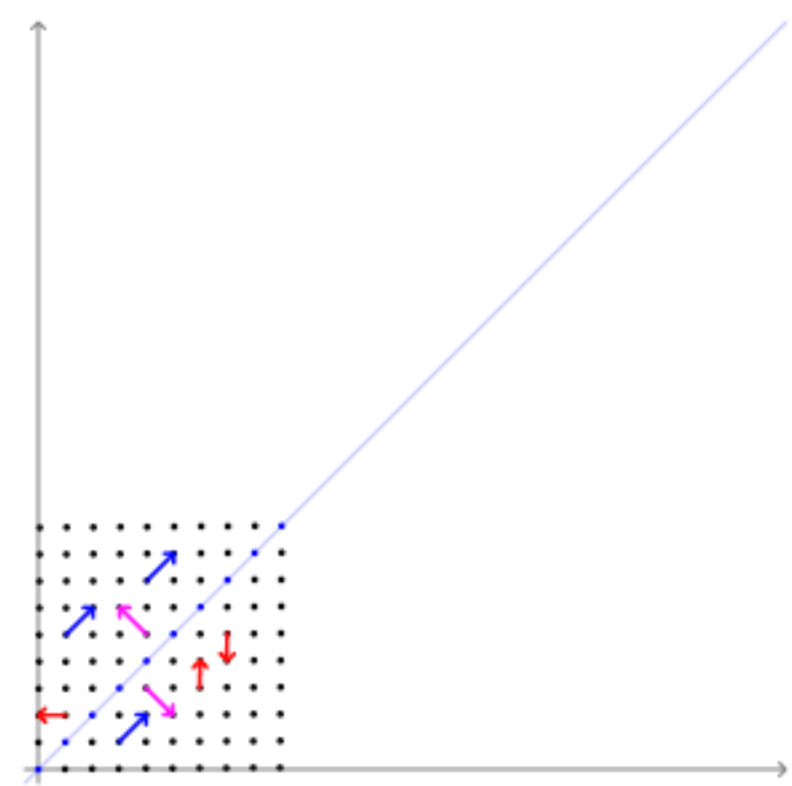

EXAMPLE: FIRST ORDER DISTORTION OF A CIRCULAR SOURCE

$$\beta_1^2 + \beta_2^2 = \beta^2$$

In the reference frame where A is diagonal:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma & 0 \\ 0 & 1-\kappa+\gamma \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\begin{aligned} \beta_1 &= (1-\kappa-\gamma)\theta_1 \\ \beta_2 &= (1-\kappa+\gamma)\theta_2 \end{aligned}$$

$$\beta^2 = \beta_1^2 + \beta_2^2 = (1-\kappa-\gamma)^2\theta_1^2 + (1-\kappa+\gamma)^2\theta_2^2$$

This is the equation of an ellipse with semi-axes:

$$a = \frac{\beta}{1-\kappa-\gamma} \qquad b = \frac{\beta}{1-\kappa+\gamma}$$